Exact solutions for the field electron emission achieved from a flat metal using the standard Fowler–Nordheim equation with a correction factor that accounts for the electric field, the work function, and the Fermi energy of the emitter

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The author uses a transfer-matrix technique to simulate field electron emission from a flat metal. The author compares in particular the results provided by this numerical scheme with those predicted by the standard Fowler–Nordheim equation. This comparison aims at establishing the influence of different approximations introduced in the standard Fowler–Nordheim theory (in particular the use of the Jeffrey–Wentzel–Kramers–Brillouin approximation for evaluating the transmission coefficient of the surface barrier and the series expansion of this coefficient when integrating over the normal-energy distribution of the incident electrons). In addition to the field and work function considered in previous work, the author explores the dependence of the emission current on the Fermi energy of the emitter. This physical parameter, which is related to the density of free carriers in the emitter, does not appear in the final form of the standard Fowler–Nordheim equation. It is therefore discarded from most analysis of field-emission data. The author shows, however, by a series of arguments that the emission currents are affected by the Fermi energy of the emitter. The author finally establishes a correction factor to be used with the Murphy–Good expression that accounts for the field, for the work function, and for the Fermi energy of the emitter and provides the exact solution for the emission achieved from a flat metal. © 2011 American Vacuum Society. [DOI: 10.1116/1.3562965]

I. INTRODUCTION

A. Background

Field electron emission was reported as early as in 1744.1 It remains, however, a subject of actuality in fundamental science and for the development of technologies.2 In the cold-emission regime in which the thermal excitation of electrons to energies that are above the surface barrier of the emitter can be neglected, this emission process is actually due to the quantum-mechanical tunneling of electrons through the surface barrier. The work function of the emitter essentially determines the energy at which this process is taking place, while the role of the external field is essentially to reduce both the height and the width of the surface barrier. The Fermi energy of the emitter, which is related to the density of free carriers, is usually not considered as a significant parameter.

The first successful modeling of field electron emission from a flat metal is attributed to Fowler and Nordheim.3 Their analysis was restricted to a triangular barrier and important extensions that incorporate the image interaction were later developed by Murphy, Good, Young, and Müller.4–6 The current density J achieved from a flat metal is given within this model by

\[ J_{\text{FN}} = \frac{a}{F} \exp \left( -\frac{b}{F} \phi / V \right), \]

where \( a = 1.541 \times 10^{-6} \) A eV V\(^{-2}\) and \( b = 6.830 \times 10^{-9} \) eV\(^{-3/2}\). \( \phi \) is the work function of the emitter, \( v \) and \( t \) are tabulated functions that account for the image interaction (they depend on \( F \) and \( \phi \) only).3,8 This equation is referred to as the “standard Fowler–Nordheim (FN) equation.” To account for the temperature \( T \) of the emitter, \( J_{\text{FN}} \) is actually multiplied by

\[ \frac{\pi k_B T}{d} \sin \left( \frac{\pi k_B T}{d} \right), \]

where \( d = \frac{a e F}{(2 \sqrt{2} m \phi)} \) (\( e \) is the elementary positive charge, \( m \) is the mass of the electron, and \( h = \hbar / 2 \pi \) with \( \hbar \) as the constant of Planck) and \( k_B \) is Boltzmann’s constant.4,5 The temperature-dependent expression is then given by

\[ J_{\text{MG}} = \frac{\pi k_B T}{d} \sin \left( \frac{\pi k_B T}{d} \right) \times \exp \left[ -\frac{b}{F} \phi / V \right]. \]

This is the “Murphy–Good expression.”

It is noticeable that the standard Fowler–Nordheim equation does not contain any parameter that relates to the kinetic energy of the electrons incident onto the inside of the barrier. This is not compatible with the principle that the wavelength of the approaching electron (which is determined by its kinetic energy) should influence its transmission coefficient. The aim of this paper is to explore how the incident electron kinetic energy affects the transmission coefficient and hence the current density \( J \). It is convenient to begin with a description of the potential-energy structure used in this work.

B. Potential-energy structure

For the quantum-mechanical modeling of field emission, we consider a system with three regions: (i) region I (\( z < 0 \)), which stands for the metal that provides the electrons; (ii) region II (\( 0 \leq z \leq L \)), which describes the surface barrier of...
the emitter; and (iii) region III \((z=L_z)\), which stands for a constant potential-energy region across which the electrons travel. It is assumed that the anode is sufficiently far from the cathode to have a negligible impact on the field-emission process. In this context, region III is just an artificial part of this modeling whose impact on the calculated currents was made negligible by taking region II sufficiently wide. These different regions are shown in Fig. 1. The constant potential-energy level in region III is used throughout the system as the energy reference level for the total electron potential energy \(V\), for the total electron energy \(E\), for the normal component \(W\) of this total energy, and for the emitter chemical potential (or Fermi level) \(\mu\). Thus, in region III, we have \(V_{III} = 0\). We require that a difference \(\Delta\) in the electrostatic component of the electron potential energy should exist across region II; thus, the local vacuum level \(V_0\) immediately outside the emitter (at \(z=0\)) is \(V_0 = \Delta\). The constant electron potential energy in region I \((V_I)\) is given by \(V_I = V_0 - \phi - E_F\), where \(E_F\) is the Fermi energy (i.e., the kinetic energy of an electron emitter at the emitter chemical potential \(\mu\)).

The potential-energy difference \(\Delta\) is taken as created by a constant external electric field \(F\) that exists only in region II. This field extracts electrons from the emitter but is treated as positive and is given by \(F = \Delta/(eL_z)\). The field intensity \(F\) is controlled in this modeling by adapting \(L_z\), while \(\Delta\) is kept unchanged. The total electron potential energy \(V_{II}(z)\) in region II is given by \(V_{II}(z) = V_0 - eFz - (1/16\pi\varepsilon_0)|e^2/z|\) (where \(\varepsilon_0\) is the electric constant). This electron potential energy \(V_{II}(z)\) is often referred to as a “Schottky–Nordheim barrier.” The barrier is actually prevented from going to \(-\infty\) as \(z \to 0\) by using the potential energy \(V_I\) in region I as lower limit.

The calculations presented in this work were achieved by using a \(\Delta\)-value of 150 eV. This value is sufficient to have region II capture the part of the Schottky–Nordheim barrier that has a significant influence on the emission currents.

### C. Role of the Fermi energy

The Fermi energy \(E_F\) is related to the density \(n\) of free carriers in the emitter by the relation \(n = (1/3\pi^2) \times (2mE_F/h^2)^{3/2}\). One would therefore expect the emission current to depend on this electron density in an explicit way. The fact \(E_F\) does not appear in the standard Fowler–Nordheim equation actually results from a series of approximations. The assumption that the emitter is a free-electron metal [kinetic energy given by \(1/2(p_x^2 + p_y^2 + p_z^2)\), with \(p_x\), \(p_y\), and \(p_z\) as the components of the momentum] enables the current density \(J\) to be written in the form \(J = e^2V/I(W)dW\). In this expression, \(W = E - (1/2m)(p_x^2 + p_y^2)\) is the normal component of the electron energy and \(N(W) = (4\pi\hbar^2/mk^2)\ln[1 + \exp(-(W - \mu)/kT)]\) is the supply function \(N(W)dW\) represents the number of incident electrons per unit surface and per unit of time, with normal energy between \(W\) and \(W + dW\). \(D(W)\) is the transmission coefficient of the surface barrier at the normal energy \(W\). The supply function \(N(W)\) does not depend on the Fermi energy \(E_F\). Any dependence of the current density \(J\) on the Fermi energy \(E_F\) must, therefore, come from the transmission coefficient \(D(W)\). Within the standard Fowler–Nordheim theory, \(D(W)\) is, however, calculated using the simple Jeffreys–Wentzel–Kramers–Brillouin (JWBK) approximation \(D(W) = \exp[-2\sqrt{2\pi}/h\cdot f(V_B - W)^{1/2}dz]\), where \(z_1\) and \(z_2\) refer to the classical turning points of the potential barrier \(V_{II}(z)\) at the normal energy \(W\). Within this approximation, \(D(W)\) only depends on \(V_{II}(z) - W\) in the tunneling part of the barrier and is therefore also independent of \(E_F\).

### D. Objectives

It is known from previous work that the simple JWBK approximation does not provide the exact solution for the electronic transmission \(D(W)\) and that corrections in the form of an effective prefactor \(P_{eff}\) must be considered in order to match the exact quantum-mechanical result.\(^{13,14}\) Since the standard FN theory relies on this JWBK approximation, it is also necessary to include a correction factor \(\lambda_{MG}\) in the (modified) Murphy–Good expression, \(J_{MG} = \lambda_{MG} \times (\pi\kappa T/d)\sin(\pi\kappa T/d)\times a^2\phi^2F^2\exp[-b_0\phi F^2/\hbar]\), in order to match the exact result.\(^{13,15}\) Previous work only addressed the dependence of these correction factors on the electric field \(F\) and on the work function \(\phi\). We show in this work that the Fermi energy \(E_F\) also contributes significantly to these corrections and we therefore propose a correction factor \(\lambda_{MG}\) for the Murphy–Good expression that accounts for the electric field \(F\), for the work function \(\phi\), and for the Fermi energy \(E_F\) of the emitter. This paper is organized according to the following lines. In Sec. II, we present the transfer-matrix (TM) technique that enables the quantum-mechanical calculation of current densities. In Sec. III, we consider the current densities achieved when the transmission coefficient of the Schottky–Nordheim barrier is calculated using the JWBK approximation. This aims at pointing the effects of some approximations in the standard Fowler–Nordheim theory and at demonstrating that any dependence of the emission current on the Fermi energy \(E_F\) is necessarily associated with a more exact calculation of the transmission coefficient. In Sec. IV, we finally investigate the influence of...
the Fermi energy $E_F$ on the current densities one obtains when using the transfer-matrix technique for the calculation of the transmission coefficient. We prove that the emission currents actually depend on the Fermi energy $E_F$ and we establish the form of this dependence. We finally propose a correction factor $\lambda^{\text{MG}}$ that enables the (modified) Murphy–Good expression to provide the exact result for the electronic emission achieved from a flat metal.

II. METHODOLOGY

Since the potential barrier is varying along one dimension only, one can actually treat the scattering problem in the Cartesian coordinates. In order to work with a finite set of boundary states, we assume that the wave function is periodic along the $x$ and $y$ coordinates (we take a lateral periodicity $L_x$ of 10 nm along the $x$ coordinate and the same lateral periodicity $L_y$ of 10 nm along the $y$ coordinate). The boundary states in regions I and III are then defined by

$$
\Psi_{i,j}^{\text{III},\pm}(r,t) = e^{i(k_{ix}\pm k_{iy}\cdot r)}e^{\pm i(2m\hbar^2)(E-V_{\text{BM}})k_x^2-k_y^2/2\hbar} e^{-iEt/\hbar},
$$

(1)

where $k_{ix} = i(2\pi/L_x)$, $k_{iy} = j(2\pi/L_y)$, and $i = \sqrt{-1}$. $E$ refers to the total electron energy and the $\pm$ sign refers to the propagation direction relative to the $z$-axis.

The next step consists in propagating these boundary states across region II. Since the barrier is independent of $x$ and $y$, there is no coupling between states associated with different values of $i$ or $j$ and one can consider the propagation of these different states separately. The idea of the method consists in assuming that the potential energy $V_{\text{II}}(z)$ in region II varies in steps along the direction $z$. For each step $\Delta z$, the solutions of Schrödinger’s equation are simple plane waves (possibly decaying in the tunneling part of the barrier). The propagation of these boundary states across region II is then achieved by matching continuity conditions for the wave function and its derivative when going from one side of region II to the other across these different steps. One can get arbitrarily close to the exact potential barrier by letting $\Delta z \rightarrow 0$ (we took $\Delta z = 0.001$ nm).

This propagation step finally leads to a set of scattering solutions of the form

$$
\Psi_{i,j} = \Psi_{i,j}^{\text{I}} + S_{(i,j),(0,0)}^{\text{I}}\Psi_{i,j}^{\text{III},+},
$$

(2)

which corresponds to single incident states $\Psi_{i,j}^{\text{I}}$ in region I. The coefficients $S_{(i,j),(0,0)}^{\text{I}}$ provide the amplitudes of the transmitted and reflected states for the incident state $\Psi_{i,j}^{\text{I}}$ in region I. The current density provided by the metal in region I is then obtained by integrating the contributions of these different scattering solutions. Referring to previous work for technical details, the result is given by

$$
J_{\text{TM}} = \frac{1}{L_x L_y} \frac{2e}{h} \int_{z_1}^{z_2} \sum_{i,j} f(E) |S_{(i,j),(0,0)}^{\text{III},+}|^2 dE,
$$

(3)

where the summation is restricted to states that are propagative in regions I and III. In this expression, $v_{\text{III},(i,j)}^{\text{I}}$ refers to the group velocity of the incident and transmitted states. $f(E) = 1/\{1 + \exp[(E - \mu)/k_B T]\}$ is the Fermi factor, with $\mu = V_0 - \phi$ as the chemical potential. The integration in Eq. (3) is achieved using a step $\Delta E$ of 0.025 eV. A room temperature $T$ of 300 K was assumed in this work.

We finally note that the fact that there is no coupling between states associated with different $i$ or $j$ makes the matrices $S^{++}$ and $S^-$ in Eqs. (2) and (3) diagonal. One can therefore construct these diagonal elements separately and the whole procedure only involves the manipulation of scalar numbers. In contrast the transfer-matrix technique presented in previous work for the consideration of three-dimensional problems involves the manipulation of large matrices, which requires more significant computational resources. This formulation of the TM technique takes full advantage of this translational invariance. It is more straightforward to implement, requires much less computational resources, and could therefore provide more accurate results.

III. EXAMINATION OF THE CURRENT DENSITIES ACHIEVED WITHIN THE JWKB APPROXIMATION

Within the JWKB approximation, the transmission coefficient of the surface barrier is estimated by $\exp(-\sqrt{2m}/\hbar \int_{z_1}^{z_2} [V_{\text{II}}(z) - E + (h^2/4m)(k_x^2 + k_y^2)/2] dz)$, with $z_1$ and $z_2$ as the classical turning points at the normal energy $E - (h^2/2m)(k_x^2 + k_y^2)$. In order to highlight the effects of a quantum-mechanical evaluation of this transmission coefficient, it is interesting to first consider the results one would obtain from the expression

$$
J_{\text{JWKB}} = \frac{1}{L_x L_y} \frac{2e}{h} \int_{z_1}^{z_2} \sum_{i,j} f(E) \left[ - \frac{2 \sqrt{2m}}{\hbar} \int_{z_1}^{z_2} [V_{\text{II}}(z) - E + (h^2/4m)(k_x^2 + k_y^2)] dz \right] dE,
$$

(4)

in which the transmission through the surface barrier is computed from the JWKB approximation (the turning points $z_1$ and $z_2$ depend on the specific values of $i$, $j$, and $E$). Since the standard Fowler–Nordheim theory is established within the same approximation, the results obtained for $J_{\text{JWKB}}$ should actually match those provided by the Murphy–Good expression,

$$
J_{\text{MG}} = \left( \frac{\pi k_B T}{\sin(\pi k_B T/d)} \right) \times \sqrt{2} \pi \phi F^2 \exp\left[ -\frac{b\phi F(3/2)}{F} \right],
$$

if the integration over states achieved within this theory was exact. This is, however, not the case. The integration achieved within the standard FN theory relies on a series expansion of the JWKB transmission coefficient and the result of this integration is, therefore, not exact. The discrepancy is well illustrated by Fig. 2, where we represented the ratio $\lambda = J_{\text{JWKB}}/J_{\text{MG}}$ between the results provided by a numerical integration of the JWKB transmission coefficient [i.e., Eq. (4)] and the results provided by the Murphy–Good expression. The results correspond to fields $F$ that range between 1 and 10 V/nm and to work functions $\phi$ that range between 1.5 and 5 eV. The representation is restricted to fields $F$ that keep below the critical value $F_{\text{crit}}$.
decreasing the work function $\phi$ between 5 and 20 eV. For a typical work function $\phi = 10$ eV, but both results achieved within the JWKB approximation, the current density $J_{\text{JWKB}}$ is insensitive to this parameter for realistic values of the Fermi energy $E_F$ that range between 1 and 10 V/nm. The work function $\phi$ tends to increase this parameter (for tested values between 5 and 20 eV).

Figure 2 is actually the representative of the error that comes from the fact that increasing $\phi$ both tend to increase this error (errors on the order of 10% are actually achieved). The reason comes from the fact that increasing $\phi$ or decreasing $\phi$ both tend to increase the width of the normal-energy distribution of the emitted electrons. This reduces the reliability of the Taylor expansion of the transmission coefficient, which is the quantity that is actually integrated within the standard FN theory. We note finally that the results presented in Fig. 2 are independent of the particular value of the Fermi energy $E_F$. This confirms the fact that, for realistic values of $E_F$ between 5 and 20 eV, any dependence of the current density $J$ on the Fermi energy $E_F$ must necessarily be associated with a more exact evaluation of the transmission coefficient (the JWKB approximation used so far leads to results that are independent of $E_F$).

IV. EXAMINATION OF THE CURRENT DENSITIES ACHIEVED USING THE TRANSFER-MATRIX TECHNIQUE

We can now consider the current densities $J_{\text{TM}}$ obtained using the transfer-matrix technique. In contrast with the results achieved within the JWKB approximation, the current densities $J_{\text{TM}}$ given by Eq. (3) provide the exact quantum-mechanical solution for the emission achieved from a flat metal (we checked that the parameters $L_{ij}$ and $\Delta (V_0)$ are sufficiently large and the parameters $\Delta z$ and $\Delta E$ are sufficiently small to enable results with at least three significant digits). The coefficient $(\partial q_{i,j}^{\text{TM}}/\partial q_{i,j}^{\text{S+}})^2$ in Eq. (3) is the quantum-mechanical equivalent of the “transmission coefficient” for the incoming electrons in region I. It will depend on the Fermi energy $E_F$ for different reasons. Through $V_i = V_0 - \phi - E_F$, it enters indeed the definition of $q_{i,j}^{\text{TM}}$ in regions I, which affects the results obtained for the coefficients $q_{i,j}^{\text{TM}}$. The summation in Eq. (3) finally extends over the different propagative states in region I. Their number is also dependent on $E_F$. One can, therefore, rightly expect the current density $J_{\text{TM}}$ to depend on the Fermi energy $E_F$.

To demonstrate that the emission current $J_{\text{TM}}$ indeed depends on the Fermi energy $E_F$, we represented in Figs. 3 and 4 the ratio $\lambda_{\text{TM}}^{\text{MG}} = J_{\text{TM}}/J_{\text{MG}}$ between the current densities $J_{\text{TM}}$ provided by the transfer-matrix technique when considering $E_F = 5$ eV (Fig. 3) and $E_F = 15$ eV (Fig. 4) and the results of

- Fig. 2. (Color online) Ratio $\lambda = J_{\text{JWKB}}/J_{\text{MG}}$ between the current density $J_{\text{JWKB}}$ obtained at $T = 300$ K from a numerical integration of the JWKB transmission coefficient [Eq. (4) in the text] and the current density $J_{\text{MG}}$ provided by the Murphy–Good expression. The results correspond to fields $F$ that range between 1 and 10 V/nm. The work function $\phi$ ranges between 1.5 and 5 eV (upward, by increments of 0.25 eV). These results correspond to a Fermi energy $E_F$ of 10 eV, but both $J_{\text{JWKB}}$ and $J_{\text{MG}}$ turn out to be insensitive to this parameter (for tested values between 5 and 20 eV).

- Fig. 3. (Color online) Ratio $\lambda_{\text{TM}}^{\text{MG}} = J_{\text{TM}}/J_{\text{MG}}$ between the current density $J_{\text{TM}}$ obtained at $T = 300$ K from a transfer-matrix calculation [Eq. (3) in the text] and the current density $J_{\text{MG}}$ provided by the Murphy–Good expression. The results correspond to fields $F$ that range between 1 and 10 V/nm. The work function $\phi$ ranges between 1.5 and 5 eV (upward, by increments of 0.25 eV). The results correspond to a Fermi energy $E_F$ of 5 eV.

- Fig. 4. (Color online) Ratio $\lambda_{\text{TM}}^{\text{MG}} = J_{\text{TM}}/J_{\text{MG}}$ between the current density $J_{\text{TM}}$ obtained at $T = 300$ K from a transfer-matrix calculation [Eq. (3) in the text] and the current density $J_{\text{MG}}$ provided by the Murphy–Good expression. The results correspond to fields $F$ that range between 1 and 10 V/nm. The work function $\phi$ ranges between 1.5 and 5 eV (upward, by increments of 0.25 eV). The results correspond to a Fermi energy $E_F$ of 15 eV.
the Murphy–Good expression $J_{MG} = \frac{\pi k_B T/d}{\sin(\pi k_B T/d)} \times \alpha r^{-3} \phi^{-1} F^2 \exp[-b_0 \phi^{3/2} / F]$. The results correspond again to fields $F$ that range between 1 and 10 V/nm and to work functions $\phi$ that range between 1.5 and 5 eV. The quantum-mechanical results $J_{TM}$ turn out to exhibit a significant dependence on the Fermi energy $E_F$. The emission currents achieved for $E_F = 5$ eV are larger than those achieved for $E_F = 15$ eV. We also observe that the quantum-mechanical results $J_{TM}$ deviate more significantly than $J_{WKJB}$ from the Murphy–Good expression $J_{MG}$. A closer agreement between $J_{TM}$ and $J_{MG}$ is, however, achieved at low fields $F$ and at high work functions $\phi$. This corresponds indeed to conditions where the JWKB approximation provides a better estimation of the transmission coefficient. The way this transmission coefficient is integrated within the standard Fowler–Nordheim theory is also more reliable in these conditions. The fact the current densities $J_{TM}$ achieved for $E_F = 5$ eV are larger than those achieved for $E_F = 15$ eV is contrary to what one would intuitively expect since a smaller Fermi energy $E_F$ also corresponds to a smaller density $n$ of free carriers in the metal. For realistic values of $E_F$ between 5 and 20 eV, the Fermi energy $E_F$ actually only influences the current density $J_{TM}$ through the quantum-mechanical transmission coefficient $(v_{III}(i,j)/v_{I}(i,j)) |S^+_{II}(i,j)|^2$. For the conditions considered in this work, this transmission coefficient appears to be generally higher for smaller values of the Fermi energy $E_F$. For values of $E_F$ that are smaller than the typical width of the normal-energy distribution of the emitted electrons, there is a cutoff of this distribution and the emission current $J_{TM}$ will finally decrease.

This is illustrated in Fig. 5 where we represented the ratio $\lambda^{MG} = J_{TM}/J_{MG}$ achieved when considering a Fermi energy $E_F$ with values between 0 and 20 eV. This result corresponds to a typical field $F$ of 5 V/nm and a typical work function $\phi$ of 4.5 eV. For $E_F \leq 1.6$ eV, there is a cutoff of the normal-energy distribution of the emitted electrons by the potential energy $V_1$ in the emitter. We find as expected that $J_{TM} \rightarrow 0$ as $E_F \rightarrow 0$, which is the result expected since the density $n$ of free carriers also tends to zero in this limit. We note that the Murphy–Good expression $J_{MG}$ does not account for a possible cutoff of the energy distribution by the potential energy in the emitter; it is indeed established within the assumption that $V_1 \rightarrow \infty$. For $E_F > 1.6$ eV, the potential energy in the emitter is sufficiently low to let any cutoff of the normal-energy distribution of the emitted electrons have a negligible impact. The emission current depends then on $E_F$ through the quantum-mechanical transmission coefficient $(v_{III}(i,j)/v_{I}(i,j)) |S^+_{II}(i,j)|^2$ and we observe that this transmission coefficient decreases with $E_F$ for the range of parameters considered. It must be noted that the transmission coefficient $(v_{III}(i,j)/v_{I}(i,j)) |S^+_{II}(i,j)|^2$ actually relates the current densities in region III to those incident in region I. This is indeed the quantum-mechanical equivalent of the transmission coefficient defined by the JWKB approximation. It is the factor $(v_{III}(i,j)/v_{I}(i,j)) |S^+_{II}(i,j)|^2$ that actually makes the quantum-mechanical expression $J_{TM}=0.878959-0.454183 \times 10^{-1} Z+0.344853 \times 10^{-2} Z^2-0.277680 \times 10^{-3} Z^3+0.190602 \times 10^{-4} Z^4-0.651969 \times 10^{-6} Z^5$, where $Z=E_F-10$, with $E_F$ as the Fermi energy in eV. This adjustment is characterized by a mean absolute error of 1.0 $\times 10^{-4}$ and a maximal absolute error of 3.9 $\times 10^{-4}$. The entire set of $\lambda^{MG}$ data that correspond to fields $F$ between 1 and 10 V/nm, to work functions $\phi$ between 1.5 and 5 eV, and to Fermi energies $E_F$ between 5 and 20 eV can be represented by a polynomial adjustment of the form $\lambda^{MG} = J_{TM}/J_{MG} = \sum_{i=0}^{5} \sum_{j=0}^{5} a_{ijk} X^i Y^j Z^k$, where $X=F-5$ with $F$ as the field strength in V/nm, $Y=\phi-3.5$ with $\phi$ as the work function in eV, and $Z=E_F-10$ with $E_F$ as the Fermi energy in eV. The coefficients $a_{ijk}$ are provided in Table I. This expression is restricted to fields $F$ that keep below $F_{max} = 4 \pi e \phi^{3/2} / e^2$. It provides a mean absolute error of 4.6 $\times 10^{-4}$ on the exact data, with a maximal absolute error of 1.2 $\times 10^{-2}$. To give a concrete example, taking $F=4$ V/nm, $\phi=5$ eV, and $E_F = 15$ eV, we have $X=-1, Y=1.5, Z=5$, and $\lambda^{MG}=0.7837$. An application that implements this polynomial adjustment and reproduces the quantum-mechanical current densities $J_{TM}$ can be found on the author’s web page.

The expression provided for $\lambda^{MG}$ extends previous work by accounting for the Fermi energy $E_F$ in addition to the field $F$ and the work function $\phi$ already considered. It was assumed that the temperature $T$ is 300 K. The $\lambda^{MG}$ values are not particularly

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**Fig. 5.** (Color online) Ratio $\lambda^{MG} = J_{TM}/J_{MG}$ between the current density $J_{TM}$ obtained at $T=300$ K from a transfer-matrix calculation [Eq. (3) in the text] and the current density $J_{MG}$ provided by the Murphy–Good expression. The results correspond to a field $F$ of 5 V/nm and to a work function $\phi$ of 4.5 eV. They are presented as a function of the Fermi energy $E_F$ of the emitter.
sensitive to this parameter and one can therefore use the expression provided for $\lambda^{\text{MG}}$ for temperatures that are close to 300 K.

V. CONCLUSION

We used a transfer-matrix technique to simulate field electron emission from a flat metal. The objective was to confront the results of an exact quantum-mechanical scheme with those provided by the standard Fowler–Nordheim equation. We investigated in particular the dependence of the emission current on the Fermi energy $E_F$ of the emitter. This parameter, which is related to the density $n$ of free carriers in the emitter, does not appear in the standard Fowler–Nordheim equation. Within the usual free-electron description of the emitter and as soon as $E_F$ exceeds the typical width of the normal-energy distribution of the emitted electrons, it turns out that the Fermi energy $E_F$ only influences the emission currents trough the probability that incident electrons have to cross the surface barrier of the emitter.

Within the standard Fowler–Nordheim theory, the transmission coefficient of the surface barrier is calculated using the simple JWKB approximation, which does not account for the particular value of $E_F$. This dependence, however, appears when an exact quantum-mechanical scheme is used and we established the form of this dependence. We finally proposed a polynomial adjustment for the correction factor $\lambda^{\text{MG}}$ to use with the (modified) Murphy–Good expression in order to match the exact quantum-mechanical result. This correction factor accounts for the field, for the work function, and for the Fermi energy of the emitter. It should be useful for the analysis of field-emission data.

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4 E. L. Murphy and R. H. Good, Phys. Rev. 102, 1464 (1956).


