The port-Hamiltonian approach to physical system modeling and control

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Part I : Network Modeling and Analysis

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From passive systems to port-Hamiltonian systems

A square nonlinear system

\[ \dot{x} = f(x) + g(x)u, \quad u \in \mathbb{R}^m \]

\[ \Sigma : \]

\[ y = h(x), \quad y \in \mathbb{R}^m \]

where \( x \in \mathbb{R}^n \) are coordinates for an \( n \)-dimensional state space \( \mathcal{X} \), is passive if there exists a storage function \( H : \mathcal{X} \rightarrow \mathbb{R} \) with \( H(x) \geq 0 \) for every \( x \), such that

\[ H(x(t_2)) - H(x(t_1)) \leq \int_{t_1}^{t_2} u^T(t)y(t)dt \]

for all solutions \((u(\cdot), x(\cdot), y(\cdot))\) and times \( t_1 \leq t_2 \).

The system is lossless if \( \leq \) is replaced by \( = \).
If $H$ is differentiable then 'passive' is equivalent to
\[
\frac{d}{dt} H \leq u^T y
\]
which reduces to (Willems, Hill-Moylan)
\[
\frac{\partial^T H}{\partial x}(x) f(x) \leq 0
\]
while in the lossless case $\leq$ is replaced by $=$.

In the linear case
\[
\dot{x} = Ax + Bu \\
y = Cx
\]
is passive if there exists a quadratic storage function $H(x) = \frac{1}{2} x^T Q x$, with $Q = Q^T \geq 0$ satisfying the LMIs
\[
A^T Q + QA \leq 0, \quad C = B^T Q
\]
Every *linear* passive system with storage function \( H(x) = \frac{1}{2} x^T Q x \), satisfying
\[
\ker Q \subset \ker A
\]
can be rewritten as a linear **port-Hamiltonian system**
\[
\begin{align*}
\dot{x} &= (J - R)Qx + Bu, \\
y &= B^T Q x,
\end{align*}
\]
\[J = -J^T, \quad R = R^T \geq 0\]

in which case the storage function \( H(x) = \frac{1}{2} x^T Q x \) is called the **Hamiltonian**.

- **Passive linear systems are thus port-Hamiltonian with non-negative Hamiltonian.**
- **Conversely** every port-Hamiltonian system with non-negative Hamiltonian is passive.
'most' nonlinear lossless systems can be written as a port-Hamiltonian system

\[
\dot{x} = J(x) \frac{\partial H}{\partial x}(x) + g(x)u
\]

\[
y = g^T(x) \frac{\partial H}{\partial x}(x)
\]

with \( J(x) = -J^T(x) \) and \( \frac{\partial H}{\partial x}(x) \) the column vector of partial derivatives. Note that

\[
\dot{x} = J(x) \frac{\partial H}{\partial x}(x)
\]

is the internal Hamiltonian dynamics known from physics, which in classical mechanics can be written as

\[
\dot{q} = \frac{\partial H}{\partial p}(q, p)
\]

\[
\dot{p} = -\frac{\partial H}{\partial q}(q, p)
\]

with the Hamiltonian \( H \) the total (kinetic + potential) energy.
Similarly, most nonlinear passive systems can be written as a port-Hamiltonian system (with dissipation)

\[ \dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u \]

\[ y = g^T(x) \frac{\partial H}{\partial x}(x) \]

with \( R(x) = R^T(x) \geq 0 \) specifying the energy dissipation

\[ \frac{d}{dt} H = - \frac{\partial^T H}{\partial x}(x) R(x) \frac{\partial H}{\partial x}(x) + u^T y \leq u^T y \]
However, in network modeling it is the other way around: one derives the system in port-Hamiltonian form (and if the Hamiltonian $H \geq 0$ then it is the storage function of a passive system).

**The matrix** $J(x)$ corresponds to the internal **power-conserving interconnection** structure of physical systems due to:

- Basic conservation laws such as Kirchhoff’s laws.
- Powerless constraints; kinematic constraints.
- Transformers, gyrators, exchange between different types of energy.

**The matrix** $R(x)$ corresponds to the internal **energy dissipation** in the system (due to resistors, damping, viscosity, etc.)

**Main message:** start with port-Hamiltonian models instead of passive models. Closer to physical modeling, and capturing more information than just the energy-balance of passivity.
A bit of port-based network modeling

The **passivity** framework considers a system component, and its power-exchange with other system components:

$$\frac{d}{dt} H \leq u^T y$$

The feedback interconnection of two passive systems

$$\frac{d}{dt} H_1 \leq u_1^T y_1, \quad \frac{d}{dt} H_2 \leq u_2^T y_2$$

$$u_1 = -y_2 + v_1, \quad u_2 = y_1 + v_2$$

leads to an interconnected system that is **again passive**, since

$$\frac{d}{dt} (H_1 + H_2) \leq u_1^T y_1 + u_2^T y_2 = v_1^T y_1 + v_2^T y_2$$
The feedback interconnection is a typical example of a *power-conserving interconnection* (total power is zero):

\[ u_1^T y_1 + u_2^T y_2 + v_1^T y_1 + v_2^T y_2 = 0 \]

In port-based modeling, e.g. bond graphs, one looks at the system as the power-conserving interconnection of ideal basic system components: (energy-) storage elements, resistive elements, transformers, gyrators, constraints, etc.

- The Hamiltonian of the resulting port-Hamiltonian system is the sum of the energies of the storage elements.
- The \( J \)- and \( B \)-matrix is determined by the transformers, gyrators, constraints, and the power-conserving interconnection.
- The \( R \)-matrix is determined by the resistive elements, and the way they are connected.
An $k$ dimensional **storage element** is determined by a $k$-dimensional state vector $x = (x_1, \cdots, x_k)$ and a Hamiltonian $H(x_1, \cdots, x_k)$ (energy storage), defining the lossless system

$$
\dot{x}_i = f_i, \quad i = 1, \cdots, k
$$

$$
e_i = \frac{\partial H}{\partial x_i}(x_1, \cdots, x_k)
$$

$$
\frac{d}{dt} H = \sum_{i=1}^{k} f_i e_i
$$

Such a $k$-dimensional storage component is written in vector notation as a port-Hamiltonian system with $J = 0$, $R = 0$, and $B = I$:

$$
\dot{x} = f
$$

$$
e = \frac{\partial H}{\partial x}(x)
$$

The elements of $x$ are called **energy variables**, those of $\frac{\partial H}{\partial x}(x)$ **co-energy variables**. Furthermore the elements of $f$ are **flow variables**, and of $e$ **effort variables**.

Note that $e^T f$ is power.
Example: The ubiquitous mass-spring-damper system:

Two storage elements:

- **Spring** Hamiltonian $H_s(q) = \frac{1}{2}kq^2$ (potential energy)

  \[
  \dot{q} = f_s \quad = \text{velocity} \\
  e_s = \frac{dH_s}{dq}(q) = kq \quad = \text{force}
  \]

- **Mass** Hamiltonian $H_m(p) = \frac{1}{2m}p^2$ (kinetic energy)

  \[
  \dot{p} = f_m \quad = \text{force} \\
  e_m = \frac{dH_m}{dp}(p) = \frac{p}{m} \quad = \text{velocity}
  \]
interconnected by

\[ f_s = e_m = y, \quad f_m = -e_s + u \]

(power-conserving since \( f_se_s + f_me_m = uy \)) yields the port-Hamiltonian system

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial q}(q,p) \\
\frac{\partial H}{\partial p}(q,p)
\end{bmatrix}
(q,p) + 
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

\[
y = 
\begin{bmatrix}
0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial q}(q,p) \\
\frac{\partial H}{\partial p}(q,p)
\end{bmatrix}
\]

with

\[ H(q,p) = H_s(q) + H_m(p) \]
Addition of the damper

\[ e_d = \frac{dR}{df_d} = c f_d, \quad R(f_d) = \frac{1}{2} c f_d^2 \]  
(Rayleigh function)

via the extended interconnection

\[ f_s = e_m = f_d = y, \quad f_m = e_s - e_d + u \]

leads to the **mass-damper-spring system** in port-Hamiltonian form

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} - \begin{bmatrix}
0 & 0 \\
0 & c
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial q}(q,p) \\
\frac{\partial H}{\partial p}(q,p)
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u
\]

\[ y = \begin{bmatrix}
0 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial q}(q,p) \\
\frac{\partial H}{\partial p}(q,p)
\end{bmatrix} \]
Example: Electro-mechanical systems

\[
\begin{bmatrix}
\dot{q} \\
\dot{p} \\
\dot{\phi}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & -\frac{1}{R}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial q}(q, p, \phi) \\
\frac{\partial H}{\partial p}(q, p, \phi) \\
\frac{\partial H}{\partial \phi}(q, p, \phi)
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} V,
\quad I = \frac{\partial H}{\partial \varphi}(q, p, \phi)
\]

Coupling electrical/mechanical domain via Hamiltonian \( H(q, p, \phi) \).
Example: LC circuits. Two inductors with magnetic energies $H_1(\varphi_1), H_2(\varphi_2)$ ($\varphi_1$ and $\varphi_2$ magnetic flux linkages), and capacitor with electric energy $H_3(Q)$ ($Q$ charge). $V$ denotes the voltage of the source.

Question: How to write this circuit as a port-Hamiltonian system in a modular way?
Hamiltonian equations for the components of the LC-circuit:

**Inductor 1**

\[ \dot{\varphi}_1 = f_1 \quad \text{(voltage)} \]

\[ e_1 = \frac{\partial H_1}{\partial \varphi_1} \quad \text{(current)} \]

**Inductor 2**

\[ \dot{\varphi}_2 = f_2 \quad \text{(voltage)} \]

\[ e_2 = \frac{\partial H_2}{\partial \varphi_2} \quad \text{(current)} \]

**Capacitor**

\[ \dot{Q} = f_3 \quad \text{(current)} \]

\[ e_3 = \frac{\partial H_3}{\partial Q} \quad \text{(voltage)} \]

All are port-Hamiltonian systems with \( J = 0 \) and \( g = 1 \).

If the elements are *linear* then the Hamiltonians are *quadratic*, e.g.

\[ H_1(\varphi_1) = \frac{1}{2L_1} \varphi_1^2, \quad \text{and} \quad \frac{\partial H_1}{\partial \varphi_1} = \frac{\varphi_1}{L_1} = \text{current}, \text{etc.} \]
Kirchhoff’s interconnection laws in $f_1, f_2, f_3, e_1, e_2, e_3, f = V, e = I$ are

\[
\begin{bmatrix}
-f_1 \\
-f_2 \\
-f_3 \\
e
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & -1 \\
0 & 0 & -1 & 0 \\
-1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
e_3 \\
f
\end{bmatrix}
\]

Substitution of eqns. of components yields port-Hamiltonian system

\[
\begin{bmatrix}
\dot{\phi}_1 \\
\dot{\phi}_2 \\
\dot{Q}
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & -1 \\
0 & 0 & 1 \\
1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial \phi_1} \\
\frac{\partial H}{\partial \phi_2} \\
\frac{\partial H}{\partial Q}
\end{bmatrix} +
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
f
\]

\[
e = \frac{\partial H}{\partial \phi_1}
\]

with $H(\phi_1, \phi_2, Q) := H_1(\phi_1) + H_2(\phi_2) + H_3(Q)$ total energy.
However, this class of port-Hamiltonian systems is **not closed under interconnection**: 

![Diagram of capacitors and inductors swapped](#)

**Figure 1: Capacitors and inductors swapped**

Interconnection leads to **algebraic constraints** between the state variables $Q_1$ and $Q_2$. 
How to model DAEs as port-Hamiltonian systems?

Intermezzo: what is the appropriate generalization of the skew-symmetric mapping $J$? Answer: Dirac structures.

('From skew-symmetric mappings to skew-symmetric relations')

Power is defined by

$$P = e(f) =: < e \mid f > = e^T f, \quad (f, e) \in \mathcal{V} \times \mathcal{V}^*.$$ 

where the linear space $\mathcal{V}$ is called the space of flows $f$ (e.g. currents), and $\mathcal{V}^*$ the space of efforts $e$ (e.g. voltages).

Symmetrized form of power is the indefinite bilinear form $\ll, \gg$ on $\mathcal{V} \times \mathcal{V}^*$:

$$\ll(f^a, e^a), (f^b, e^b) \gg := < e^a \mid f^b > + < e^b \mid f^a >,$$

$$(f^a, e^a), (f^b, e^b) \in \mathcal{V} \times \mathcal{V}^*.$$
Definition 1 (Weinstein, Courant, Dorfman) A (constant) Dirac structure is a subspace
\[ D \subset \mathcal{V} \times \mathcal{V}^* \]
such that
\[ D = D^\perp, \]
where \( \perp \) denotes orthogonal complement with respect to the bilinear form \( \langle \cdot, \cdot \rangle \).

For a finite-dimensional space \( \mathcal{V} \) this is equivalent to
(i) \( \langle e \mid f \rangle e^T f = 0 \) for all \((f, e) \in D\),
(ii) \( \dim D = \dim \mathcal{V} \).

For any skew-symmetric map \( J : \mathcal{V}^* \rightarrow \mathcal{V} \) its graph
\[ \{(f, e) \in \mathcal{V} \times \mathcal{V}^* \mid f = Je\} \]
is a Dirac structure!
For many systems, especially those with 3-D mechanical components, the interconnection structure will be *modulated* by the energy or geometric variables.

This leads to the notion of *non-constant* Dirac structures on *manifolds*.

**Definition 2** Consider a smooth manifold $M$. A *Dirac structure on* $M$ *is a vector subbundle* $\mathcal{D} \subset TM \oplus T^*M$ *such that for every* $x \in M$ *the vector space* 

$$\mathcal{D}(x) \subset T_xM \times T^*_xM$$

*is a Dirac structure as before.*
Geometric definition of a port-Hamiltonian system

\[
\begin{align*}
\dot{x} & = \frac{\partial H}{\partial x}(x) \\
D(x) & = \{ f(t), e(t) \} \in \mathcal{D}(x(t)), \quad t \in \mathbb{R}
\end{align*}
\]

Figure 2: Port-Hamiltonian system

The dynamical system defined by the DAEs

\[
\begin{align*}
-\dot{x}(t) &= f_x(t), \\
\frac{\partial H}{\partial x}(x(t)) &= e_x(t), \quad t \in \mathbb{R}
\end{align*}
\]

is called a port-Hamiltonian system.
Particular case is a Dirac structure $\mathcal{D}(x) \subset T_x\mathcal{X} \times T^*_x\mathcal{X} \times \mathcal{F} \times \mathcal{F}^*$ given as the graph of the skew-symmetric map

\[
\begin{bmatrix}
    f_x \\
    e
\end{bmatrix} = \begin{bmatrix}
    -J(x) & -g(x) \\
    g^T(x) & 0
\end{bmatrix} \begin{bmatrix}
    e_x \\
    f
\end{bmatrix},
\]

leading ($f_x = -\dot{x}$, $e_x = \frac{\partial H}{\partial x}(x)$) to a port-Hamiltonian system as before

\[
\dot{x} = J(x) \frac{\partial H}{\partial x}(x) + g(x)f, \quad x \in \mathcal{X}, f \in \mathbb{R}^m
\]

\[
e = g^T(x) \frac{\partial H}{\partial x}(x), \quad e \in \mathbb{R}^m
\]
Energy-dissipation is included by terminating some of the ports by static resistive elements

\[ f_R = -F(e_R), \quad \text{where } e_R^T F(e_R) \geq 0, \quad \text{for all } e_R. \]

If \( H \geq 0 \) then the system stays passive with respect to the remaining flows and efforts \( f \) and \( e \):

\[ \frac{d}{dt} H \leq e^T f \]

This leads, e.g. for linear damping, to input-state-output port-Hamiltonian systems in the form

\[
\begin{aligned}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)f \\
e &= g^T(x) \frac{\partial H}{\partial x}(x)
\end{aligned}
\]

where \( J(x) = -J^T(x), \ R(x) = R^T(x) \geq 0 \) are the interconnection and damping matrices, respectively.
Example: Mechanical systems with kinematic constraints

Constraints on the generalized velocities $\dot{q}$:

$$A^T(q)\dot{q} = 0.$$ 

This leads to constrained Hamiltonian equations

$$\dot{q} = \frac{\partial H}{\partial p}(q, p)$$
$$\dot{p} = -\frac{\partial H}{\partial q}(q, p) + A(q)\lambda + B(q)u$$
$$0 = A^T(q)\frac{\partial H}{\partial p}(q, p)$$
$$y = B^T(q)\frac{\partial H}{\partial p}(q, p)$$

with $H(q, p)$ total energy, and $\lambda$ the constraint forces.
By elimination of the constraints and constraint forces one derives a port-Hamiltonian model without constraints.

Can be extended to general multi-body systems.
Intermezzo: Relation with classical Hamiltonian equations

\[ \dot{x} = J \frac{\partial H}{\partial x}(x) \]

with constant or 'integrable' \(J\)-matrix admits coordinates \(x = (q, p, r)\) in which

\[
J = \begin{bmatrix}
0 & I & 0 \\
-I & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}, \quad \dot{q} = \frac{\partial H}{\partial p}(q, p, r) \]
\[
\dot{p} = -\frac{\partial H}{\partial q}(q, p, r) \]
\[
\dot{r} = 0
\]

For constant or integrable Dirac structure one gets Hamiltonian DAEs

\[
\dot{q} = \frac{\partial H}{\partial p}(q, p, r, s) \]
\[
\dot{p} = -\frac{\partial H}{\partial q}(q, p, r, s) \]
\[
\dot{r} = 0 \]
\[
0 = \frac{\partial H}{\partial s}(q, p, r, s)
\]
Some properties of port-Hamiltonian systems

Port-Hamiltonian systems modeling encodes more information than energy-balance.

The Dirac structure determines all the Casimir functions (conserved quantities which are independent of $H$).

**Example**: In the first LC circuit the total flux $\phi_1 + \phi_2$ is a conserved quantity that is solely determined by the circuit topology. (In Part II this will be used for set-point control.)

Furthermore, the Dirac structure directly determines the algebraic constraints.

**Example**: In the second LC-circuit the state variables $Q_1$ and $Q_2$ are related by

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$
Any power-conserving interconnection of port-Hamiltonian systems is again port-Hamiltonian

- The resulting Hamiltonian is the sum of the Hamiltonians of the individual systems.
- The Dirac structure is determined by the Dirac structures of the individual systems, and the way they are interconnected.
- The resistive structure is determined by the resistive structures of the individual systems.

Conclusion: port-Hamiltonian systems theory provides a modular framework for modeling and analysis of complex multi-physics lumped-parameter systems.
Network modeling is prevailing in modeling and simulation of lumped-parameter physical systems (multi-body systems, electrical circuits, electro-mechanical systems, hydraulic systems, robotic systems, etc.), with many advantages:

- Modularity and flexibility. Re-usability (‘libraries’).
- Multi-physics approach.
- Suited to design/control.

*Disadvantage* of network modeling: it generally leads to a large set of DAEs, *seemingly without any structure.*

Port-based modeling and port-Hamiltonian system theory identifies the underlying structure of network models of physical systems, to be used for analysis, simulation and control.
Distributed-parameter port-Hamiltonian systems

Figure 3: Simplest example: Transmission line

Telegrapher’s equations define the boundary control system

\[
\begin{align*}
\frac{\partial Q}{\partial t}(z,t) & = -\frac{\partial}{\partial z}I(z,t) & = -\frac{\partial}{\partial z} \frac{\Phi(z,t)}{L(z)} \\
\frac{\partial \Phi}{\partial t}(z,t) & = -\frac{\partial}{\partial z}V(z,t) & = -\frac{\partial}{\partial z} \frac{Q(z,t)}{C(z)} \\
f_a(t) & = V(a,t), \quad e_1(t) = I(a,t) \\
f_b(t) & = V(b,t), \quad e_2(t) = I(b,t)
\end{align*}
\]
Transmission line as port-Hamiltonian system

Define *internal* flows \( f_x = (f_E, f_M) \) and efforts \( e_x = (e_E, e_M) \):

- **electric flow** \( f_E : [a, b] \to \mathbb{R} \)
- **magnetic flow** \( f_M : [a, b] \to \mathbb{R} \)
- **electric effort** \( e_E : [a, b] \to \mathbb{R} \)
- **magnetic effort** \( e_M : [a, b] \to \mathbb{R} \)

Togeter with *external* boundary flows \( f = (f_a, f_b) \) and boundary efforts \( e = (e_a, e_b) \). Define the *infinite-dimensional Dirac structure*

\[
\begin{bmatrix}
  f_E \\
  f_M
\end{bmatrix} = \begin{bmatrix}
  0 & \frac{\partial}{\partial z} \\
  \frac{\partial}{\partial z} & 0
\end{bmatrix}
\begin{bmatrix}
  e_E \\
  e_M
\end{bmatrix}
\]

\[
\begin{bmatrix}
  f_{a,b} \\
  e_{a,b}
\end{bmatrix} = \begin{bmatrix}
  e_{E|a,b} \\
  e_{M|a,b}
\end{bmatrix}
\]
This defines a Dirac structure on the space of *internal* flows and efforts and *boundary* flows and efforts.

Substituting (as in the lumped-parameter case)

\[
\begin{align*}
  f_E &= -\frac{\partial Q}{\partial t} \\
  f_M &= -\frac{\partial \varphi}{\partial t} \\
  f_x &= -\dot{x}
\end{align*}
\]

\[
\begin{align*}
  e_E &= \frac{Q}{C} = \frac{\partial H}{\partial Q} \\
  e_M &= \frac{\varphi}{L} = \frac{\partial H}{\partial \varphi} \\
  e_x &= \frac{\partial H}{\partial x}
\end{align*}
\]

with, for example, quadratic energy density

\[
H(Q, \varphi) = \frac{1}{2} \frac{Q^2}{C} + \frac{1}{2} \frac{\varphi^2}{L}
\]

we recover the telegrapher’s equations.
Of course, the telegrapher’s equations can be rewritten as the linear wave equation

\[
\frac{\partial^2 Q}{\partial t^2} = - \frac{\partial}{\partial z} \frac{\partial I}{\partial t} = - \frac{\partial}{\partial z} \frac{\partial \phi}{\partial t} = - \frac{\partial}{\partial z} \frac{1}{L} \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial z} \frac{1}{L} \frac{\partial Q}{\partial z} = \frac{1}{LC} \frac{\partial^2 Q}{\partial z^2}
\]

(if \(L(z), C(z)\) do not depend on \(z\)), or similar expressions in \(\phi, I\) or \(V\).

The same equations hold for a vibrating string, or for a compressible gas/fluid in a one-dimensional pipe.

**Basic question:**

Which of the boundary variables \(f_a, f_b, e_a, e_b\) can be considered to be inputs, and which outputs? See Lecture of Hans Zwart.
Example 2: Shallow water equations; distributed-parameter port-Hamiltonian system with non-quadratic Hamiltonian

The dynamics of the water in an open-channel canal can be described by

\[
\begin{align*}
\partial_t \begin{bmatrix} h \\ v \end{bmatrix} + \begin{bmatrix} v & h \\ g & v \end{bmatrix} \partial_z \begin{bmatrix} h \\ v \end{bmatrix} &= 0
\end{align*}
\]

with \( h(z,t) \) the height of the water at position \( z \), and \( v(z,t) \) the velocity (and \( g \) gravitational constant).

This can be written as a port-Hamiltonian system by recognizing the total energy

\[
H(h,v) = \frac{1}{2} \int_a^b [hv^2 + gh^2] \, dz
\]
yielding the co-energy functions\(^a\)

\[
e_h = \frac{\partial H}{\partial h} = \frac{1}{2} v^2 + gh \quad \text{Bernoulli function}
\]

\[
e_v = \frac{\partial H}{\partial v} = hv \quad \text{mass flow}
\]

It follows that the shallow water equations can be written, similarly to the telegraphers equations, as

\[
\frac{\partial h}{\partial t}(z,t) = -\frac{\partial}{\partial z} \frac{\partial H}{\partial v}
\]

\[
\frac{\partial v}{\partial t}(z,t) = -\frac{\partial}{\partial z} \frac{\partial H}{\partial h}
\]

with boundary variables \(-hv\big|_{a,b}\) and \(((\frac{1}{2}v^2 + gh)\big|_{a,b})\).

\(^a\)Daniel Bernoulli, born in 1700 in Groningen as son of Johann Bernoulli, professor in mathematics at the University of Groningen and forerunner of the Calculus of Variations (the Brachistochrone problem).
Paying tribute to history:

Figure 4: Johann Bernoulli, professor in Groningen 1695-1705.

Figure 5: Daniel Bernoulli, born in Groningen in 1700.
We obtain the energy balance

\[
\frac{d}{dt} \int_a^b [hv^2 + gh^2] \, dz = -(hv)(\frac{1}{2}v^2 + gh)|_a^b
\]

which can be rewritten as

\[
-v(\frac{1}{2}gh^2)|_a^b - v(\frac{1}{2}hv^2 + \frac{1}{2}gh^2))|_a^b =
\]

velocity \times pressure + energy flux through the boundary
Conservation laws

All examples so far have the same structure

$$\frac{\partial \alpha_1}{\partial t}(z,t) = - \frac{\partial}{\partial z} \frac{\partial H}{\partial \alpha_2} = - \frac{\partial}{\partial z} \beta_2$$

$$\frac{\partial \alpha_2}{\partial t}(z,t) = - \frac{\partial}{\partial z} \frac{\partial H}{\partial \alpha_1} = - \frac{\partial}{\partial z} \beta_1$$

with boundary variables $\beta_1|_{\{a,b\}}, \beta_2|_{\{a,b\}}$, corresponding to two coupled conservation laws:

$$\frac{d}{dt} \int_a^b \alpha_1 = - \int_a^b \frac{\partial}{\partial z} \beta_2 = \beta_2(a) - \beta_2(b)$$

$$\frac{d}{dt} \int_a^b \alpha_2 = - \int_a^b \frac{\partial}{\partial z} \beta_1 = \beta_1(a) - \beta_1(b)$$

(In the transmission line, $\alpha_1$ and $\alpha_2$ is charge- and flux-density, and $\beta_1, \beta_2$ voltage $V$ and current $I$, respectively.)
For some purposes it is illuminating to rewrite the equations in terms of the co-energy variables $\beta_1, \beta_2$:

$$\begin{bmatrix}
\frac{\partial \beta_1}{\partial t} \\
\frac{\partial \beta_2}{\partial t}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial^2 H}{\partial \alpha_1^2} & \frac{\partial^2 H}{\partial \alpha_1 \alpha_2} \\
\frac{\partial^2 H}{\partial \alpha_2 \alpha_1} & \frac{\partial^2 H}{\partial \alpha_2^2}
\end{bmatrix} \begin{bmatrix}
\frac{\partial \alpha_1}{\partial t} \\
\frac{\partial \alpha_2}{\partial t}
\end{bmatrix} = - \begin{bmatrix}
\frac{\partial^2 H}{\partial \alpha_1^2} & \frac{\partial^2 H}{\partial \alpha_1 \alpha_2} \\
\frac{\partial^2 H}{\partial \alpha_2 \alpha_1} & \frac{\partial^2 H}{\partial \alpha_2^2}
\end{bmatrix} \begin{bmatrix}
\frac{\partial \beta_1}{\partial z} \\
\frac{\partial \beta_2}{\partial z}
\end{bmatrix}$$

For the transmission line this yields

$$\begin{bmatrix}
\frac{\partial V}{\partial t} \\
\frac{\partial I}{\partial t}
\end{bmatrix} = - \begin{bmatrix}
0 & \frac{1}{C} \\
\frac{1}{L} & 0
\end{bmatrix} \begin{bmatrix}
\frac{\partial V}{\partial z} \\
\frac{\partial I}{\partial z}
\end{bmatrix}$$

The matrix is called the **characteristic matrix**, whose eigenvalues are the characteristic velocities $\frac{1}{\sqrt{LC}}$ and $-\frac{1}{\sqrt{LC}}$ corresponding to the characteristic eigenvectors (and curves).
For the shallow water equations this yields

\[
\begin{bmatrix}
\frac{\partial \beta_1}{\partial t} \\
\frac{\partial \beta_2}{\partial t}
\end{bmatrix}
= -
\begin{bmatrix}
v & g \\
h & v
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \beta_1}{\partial z} \\
\frac{\partial \beta_2}{\partial z}
\end{bmatrix}
\]

with

\[
\beta_1 = \frac{1}{2} v^2 + gh, \quad \beta_2 = hv
\]

being the Bernoulli function and mass flow, respectively.

This corresponds to two characteristic velocities \( v \pm \sqrt{gh} \), which are, like in the transmission line case, of opposite sign (\textit{subcritical} or \textit{fluvial} flow) if

\[
v^2 \leq gh
\]

Because the Hamiltonian is non-quadratic, and thus the pde’s are nonlinear, the characteristic curves may \textbf{intersect}, corresponding to shock waves.
Mixed lumped- and distributed-parameter port-Hamiltonian systems

Typical example: power-converter connected via a transmission line to a resistive load or an induction motor:

- The power-converter is a port-Hamiltonian system (with switching Dirac structure).

- Transmission line is distributed-parameter port-Hamiltonian system.

- Induction motor is a port-Hamiltonian system, with Hamiltonian being the electro-mechanical energy.
Power converter connected to the load via transmission line

Figure 6: The Boost converter with a transmission line
Boost power converter

The circuit consists of a capacitor $C$ with electric charge $q_C$, an inductor $L$ with magnetic flux linkage $\phi_L$, and a resistive load $R$, together with an ideal diode and an ideal switch $S$, with switch positions $s = 1$ (switch closed) and $s = 0$ (switch open).
The voltage-current characteristics of the ideal diode and switch are depicted in Figure 8.

![Diode and Switch Characteristic Diagram]

**Figure 8**: Voltage-current characteristic of an ideal diode and ideal switch

The ideal diode thus satisfies the complementarity conditions:

\[ v_D i_D = 0, \quad v_D \leq 0, \quad i_D \geq 0. \]
This yields the port-Hamiltonian model (with \( H(q_C, \phi_L) = \frac{1}{2C}q_C^2 + \frac{1}{2L}\phi_L^2 \)):

\[
\begin{bmatrix}
\dot{q}_C \\
\dot{\phi}_L
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{R} & 1 - s \\
 s - 1 & 0
\end{bmatrix} \begin{bmatrix}
\frac{\partial H}{\partial q_C} = \frac{q_C}{C} \\
\frac{\partial H}{\partial \phi_L} = \frac{\phi_L}{L}
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} E + \begin{bmatrix}
si_D \\
(s - 1)v_D
\end{bmatrix}
\]

\[
I = \frac{\phi_L}{L}
\]

Assume that the switch and the diode are coupled in the following sense: if the switch is closed \((s = 1)\) then the diode is open \((i_D = 0)\), while if the switch is open \((s = 0)\), then the diode is closed \((v_D = 0)\). (This means that we disregard the so-called discontinuous modes.)
In this case we obtain the switching port-Hamiltonian system

\[
\begin{bmatrix}
\dot{q}_C \\
\dot{\phi}_L
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{R} & 1 - s \\
 s - 1 & 0
\end{bmatrix} \begin{bmatrix}
\frac{\partial H}{\partial q_C} = \frac{q_C}{C} \\
\frac{\partial H}{\partial \phi_L} = \frac{\phi_L}{L}
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} E
\]

\[
I = \begin{bmatrix}
0 & 1
\end{bmatrix} \begin{bmatrix}
\frac{\partial H}{\partial q_C} = \frac{q_C}{C} \\
\frac{\partial H}{\partial \phi_L} = \frac{\phi_L}{L}
\end{bmatrix} = \frac{\phi_L}{L}
\]

A port-Hamiltonian system where the J-matrix depends on the switch position.
In general, a switching port-Hamiltonian system (without algebraic equality and inequality constraints) is defined as

\[ \dot{x} = F(\rho)z + g(\rho)u, \quad z = \frac{\partial H}{\partial x}(x) \]

\[ y = g^T(\rho)z \]

with \( \rho \in \{0, 1\}^p \), and

\[ F(\rho) = J(\rho) - R(\rho), \quad J(\rho) = -J^T(\rho), \quad R(\rho) = R^T(\rho) \geq 0 \]

Note that the system is passive for every switching sequence:

\[ \frac{d}{dt} H = -\frac{\partial^T H}{\partial x}(x)R(\rho)\frac{\partial H}{\partial x}(x) + u^T y \leq u^T y \]
Oriented graphs and Kirchhoff behavior

An oriented graph $\mathcal{G}$ consists of a finite set $\mathcal{N}$ of nodes (vertices) and a finite set $\mathcal{B}$ of branches (edges). To every branch $b \in \mathcal{B}$ there corresponds an ordered pair $(n, m) \in \mathcal{N}^2$ representing the starting node $n$ and the final node $m$ of this branch.
An oriented graph is specified by its *incidence matrix* $D$, which is an $\bar{n} \times \bar{b}$ matrix, $\bar{n}$ being the number of nodes and $\bar{b}$ being the number of branches, with $(i,j)$–th element $d_{ij}$ equal to 1 if the $j$-th branch is a branch towards node $i$, equal to $-1$ if the $j$-th branch is a branch originating from node $i$, and equal to 0 otherwise.
The node space $\Lambda_0$ is the vector space of all functions from $\mathcal{N}$ to $\mathbb{R}$. Clearly, $\Lambda_0$ can be identified with $\mathbb{R}^n$.

The branch space $\Lambda_1$ is the vector space of all functions from $\mathcal{B}$ to $\mathbb{R}$. Again, $\Lambda_1$ can be identified with $\mathbb{R}^b$.

For an electrical circuit $\Lambda_1$ will be the vector space of currents over the branches in the circuit.

The dual space of the vector space $\Lambda_1$ will be denoted by $\Lambda^1$; the vector space of voltages over the branches.

The product $<V|I>$ of a vector of currents $I \in \Lambda_1$ with a vector of voltages $V \in \Lambda^1$ is the total power over the circuit.

$\Lambda^0$ is the vector space of potentials over the nodes, with dual space $\Lambda_0$.

(Note: since $\Lambda_0$ and $\Lambda_1$ have a canonical basis we can identify them with their dual spaces $\Lambda^0$ and $\Lambda^1$.)
The incidence matrix $D$ is the matrix representation of a map

$$\partial : \Lambda_1 \to \Lambda_0$$

called the **incidence operator**$^a$. The adjoint map of the incidence operator $\partial$ is the linear map

$$d : \Lambda^0 \to \Lambda^1$$

which is called the **co-incidence** operator. The matrix representation of the map $d$ is given by $D^T$.

$^a$In the literature this operator is usually called the boundary operator.
Kirchhoff’s current laws (KCL) for a circuit are given as
\[ \partial I = 0 \]
while Kirchhoff’s voltage laws (KVL) take the form
\[ V \in \text{im} d \]
(The kernel of the incidence operator \( \partial \) is the cycle space of the graph, the image of the co-incidence operator \( d \) is its cut space.) Kirchhoff’s voltage laws are thus
\[ V = d\phi \]
for some \( \phi \in \Lambda^0 \) (the vector of potentials at every node). In matrix notation Kirchhoff’s laws are
\[ DI = 0, \quad V = D^T \phi \]
Tellegen’s theorem automatically follows from Kirchhoff’s laws.

Take any current distribution \( I \) satisfying KCL, and any voltage distribution \( V \) satisfying KVL. Then

\[
\sum_b V_b I_b = \sum_b (d\phi)_b I_b = \sum_n \phi_n (\partial I)_n = 0
\]

since \( I \) satisfies KCL \( \partial I = 0 \).

In particular, Tellegen’s theorem implies that for any actual current and voltage distribution over the circuit the total power is equal to zero.
The **Kirchhoff behavior** $B_K(G)$ of a graph $G$ with incidence operator $\partial$ is

$$B_K(G) := \{(I, V) \in \Lambda_1 \times \Lambda^1 | I \in \ker \partial, V \in \im d\}$$

It immediately follows that the Kirchhoff behavior defines a **Dirac structure**.

$D \subset V \times V^*$ is a Dirac structure if $<v^* | v> = 0$ for every $(v, v^*) \in D$ while $\dim D = \dim V$. 
Open graphs

An open graph $\mathcal{G}$ is obtained from an ordinary graph with set of nodes $\mathcal{N}$ by identifying a subset $\mathcal{N}_e \subset \mathcal{N}$ of external nodes. The remaining subset $\mathcal{N}_i := \mathcal{N} - \mathcal{N}_e$ are the internal nodes of the open graph.

Kirchhoff’s current laws now take the form

$$\partial I + \begin{bmatrix} I_e \\ 0 \end{bmatrix} = 0$$

with $I_e \in \Lambda_e$ the vector of external currents entering the graph at the external nodes (terminals).
In this case we obtain

\[ \sum_b V_b I_b = \sum_b (d\phi)_b I_b = \sum_n \phi_n (\partial I)_n = \]
\[ \sum_{n_i} \phi_{n_i} (\partial I)_{n_i} + \sum_{n_e} \phi_{n_e} (\partial I)_{n_e} = -\sum_{n_e} \phi_{n_e} I_{n_e} \] (1)

since \( I \) satisfies the Kirchhoff’s current laws \( (\partial I)_{n_i} = 0 \) and \( (\partial I)_{n_e} = -I_{n_e} \).

Thus, for open graphs the total power over the graph is **not zero** but equal to \(-\sum_{n_e} \phi_{n_e} I_{n_e}\) (the incoming power at the external nodes).
Even though the vector of potentials $\phi_e$ at the external nodes is **not** uniquely determined by the vector $V$ of voltages, the expression $\sum_{n_e} \phi_{n_e} I_{n_e}$ is uniquely determined.

The freedom in the choice of $\phi$ corresponding to the same vector of voltages $V$ is given by all vectors $\psi$ such that $D^T \psi = 0$. Hence the freedom in $\phi_e$ is given by all vectors $\psi_e$ such that for some $\psi_i$ it holds that $D_e^T \psi_e + D_i^T \psi_i = 0$, where $D_e$ is the submatrix of $D$ consisting of the first $\bar{n}_e$ rows and $D_i$ is the submatrix consisting of the last $\bar{n} - \bar{n}_e$ rows. For any such $\psi_e$ we have

$$\psi_e^T I_e = -\psi_e^T D_e I = \psi_i^T D_i I = 0$$
The image of $D$, or equivalently the kernel of $D^T$, can be characterized as follows. First

$$0 = \mathbb{1}^T DI = \sum_{n_e} I_{n_e}$$  \hspace{1cm} (2)

Hence the external part of the Kirchhoff behavior of an open graph is constrained by the obvious fact that all external currents sum up to zero.

In general, the rank of $D$ is equal to $\bar{n} - k_G$, where $k_G$ denotes the number of connected components.

For example, if $G$ consists of two connected components, then the vectors $\begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \mathbb{1} \end{bmatrix}$ span the kernel of $D^T$. This implies that both the sum of the external currents belonging to the first component as well as the sum of the external currents of the second component are equal to zero.
It turns out that an open graph $\mathcal{G}$ can be closed (in a unique fashion) to an ordinary graph $\bar{\mathcal{G}}$ while keeping the same number of connected components.

Let $\mathcal{G}$ be connected. Then add one virtual node $\bar{n}$, and virtual edges from this virtual node $\bar{n}$ to every external node $n_e \in \mathcal{N}_e$. The Kirchhoff behavior of this graph $\bar{\mathcal{G}}$ extends the Kirchhoff behavior of the open graph $\mathcal{G}$. In fact, to the virtual node $\bar{n}$ we may associate an arbitrary potential $\phi_{\bar{n}}$ (a ground-potential), and we may rewrite the righthand-side of (1) as (since $\sum_e I_{ne} = 0$)

$$-\sum_{n_e} (\phi_{n_e} - \phi_{\bar{n}}) I_{n_e} = -\sum_{n_e} V_{n_e} I_{n_e}$$

where $V_{n_e} := \phi_{n_e} - \phi_{\bar{n}}$ and $I_{n_e}$ denote the voltage, respectively current, over the virtual edge towards the external node $n_e$.

Thus the incoming power is rewritten as a product of external currents and voltages.
Constitutive relations for every branch.

Simplest case is a resistive relation between $I_b$ and $V_b$ such that $V_b I_b \geq 0$.

In the case of a capacitive relation one defines the charge $Q_b$ together with the electric energy $H_b(Q_b)$. The constitutive relations are then given by

\[ \dot{Q}_b = -I_b, \quad V_b = \frac{dH_b}{dQ_b}(Q_b) \]

Alternatively, in the case of an inductor one specifies the magnetic energy $H_b(\varphi_b)$, where $\varphi_b$ denotes the flux, together with the relations

\[ \dot{\varphi}_b = -V_b, \quad I_b = \frac{dH_b}{d\varphi_b}(\varphi_b) \]

Substituting these constitutive relations into the Kirchhoff behavior (and corresponding Dirac structure) defined by the graph results in a port-Hamiltonian system description.
Interconnection of open graphs

Consider two open graphs $G^A$ and $G^B$ with respective sets of external nodes $N^A_e$ and $N^B_e$. Suppose we want to interconnect these open graphs over a set of *shared external nodes*

$$\bar{N} \subset N^A_e \cap N^B_e$$

This may be done by identifying the shared nodes corresponding to the two graphs, leading to an interconnected open graph $G^A \parallel G^B$ with resulting set of internal nodes

$$N^A_i \cup N^B_i \cup \bar{N}$$

and external nodes

$$(N^A_e - \bar{N}) \cup (N^B_e - \bar{N})$$
Extension to higher-order networks

An oriented graph $\mathcal{G}$ with incidence operator $\partial$ defines a 1-dimensional complex. Indeed, the sequence

$$\Lambda_1 \xrightarrow{\partial} \Lambda_0 \xrightarrow{\Pi} \mathbb{R}$$

satisfies $\Pi \circ \partial = 0$.

The 'algebraic-topological invariants' of this 1-complex (the so-called Betti numbers) are nothing else than the number of connected components.
An arbitrary $k$-complex $\Lambda$ is specified by a sequence of linear spaces $\Lambda_0, \Lambda_1, \ldots, \Lambda_k$, together with a sequence of incidence operators

$$\Lambda_k \xrightarrow{\partial_k} \Lambda_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_1} \Lambda_1 \xrightarrow{\partial_0} \Lambda_0 \xrightarrow{\partial_0} 0$$

with the property that

$$\partial_{j-1} \circ \partial_j = 0, \quad j = 1, \ldots, k$$

The vector spaces $\Lambda_j, j = 0, 1 \cdots, k$, are called the spaces of $j$-chains.

Each $\Lambda_j$ is generated by a finite set of $j$-cells in the sense that $\Lambda_j$ is the set of functions from the set of $j$-cells to $\mathbb{R}$. 
A typical example of a 3-complex is the triangularization of a 3-dimensional manifold, with the $j$-cells, $j = 0, 1, 2, 3$, being the sets of vertices, edges, faces, and tetrahedra, and $\partial_1, \partial_2, \partial_3$ capturing the incidence structure.

Denoting the dual linear spaces by $\Lambda^j$, $j = 0, 1 \cdots, k$, we have the following dual sequence

$$
\Lambda^0 \xrightarrow{d_1} \Lambda^1 \xrightarrow{d_2} \Lambda^2 \cdots \Lambda^{k-1} \xrightarrow{d_k} \Lambda^k
$$

having the analogous property

$$
d_j \circ d_{j-1} = 0, \quad j = 2, \cdots, k
$$

The elements of $\Lambda^j$ are called $j$-cochains.
Consider any \( k \)-complex \( \Lambda \), with \( k \)-chains \( \alpha \in \Lambda_k \) and \( k \)-cochains \( \beta \in \Lambda^k \). We define, similarly as in the case of a graph (1-complex) its Kirchhoff behavior as

\[
B_K(\Lambda) := \{(\alpha, \beta) \in \Lambda_k \times \Lambda^k \mid \partial_k \alpha = 0, \exists \phi \in \Lambda^{k-1} \text{ s.t. } \beta = d_k \phi \}
\]

As before, it is immediately seen that \( B_K(\Lambda) \subset \Lambda_k \times \Lambda^k \) is a Dirac structure. In particular, it follows that

\[
< \beta \mid \alpha >_k = 0
\]

for every \((\alpha, \beta) \in B_K(\Lambda)\), where \(< \cdot \mid \cdot >_k\) denotes the duality product between the dual linear spaces \( \Lambda_k \) and \( \Lambda^k \).
Open $k$-complexes

An open $k$-complex is obtained by identifying a subset $\mathcal{N}_{(k-1)e}$ of the set of all $(k-1)$-cells, called the external $(k-1)$-cells.

Define the linear space of functions from this subset of $(k-1)$-cells to $\mathbb{R}$ as $\Lambda_e \subset \Lambda_{k-1}$ (the space of 'external currents').
As before, Kirchhoff’s voltage laws remain unchanged

\[ \beta = d_k \phi, \]

while Kirchhoff’s current laws are modified into

\[ \partial_k \alpha + \begin{bmatrix} \alpha_e \\ 0 \end{bmatrix} = 0 \]

where \( \alpha_e \) is the vector of external currents associated to the external \((k-1)\)-cells. We obtain

\[ < \beta | \alpha >_k = < d_k \phi | \alpha >_k = < \phi | \partial_k \alpha >_{k-1} = \]

\[ < \phi | \begin{bmatrix} -\alpha_e \\ 0 \end{bmatrix} >_{k-1} = - < \phi_e | \alpha_e >_{k-1} \]

where \( \phi_e \) denotes the vector of potentials at the external \((k-1)\)-cells.
Similar to graphs it follows that the Kirchhoff current laws for open $k$-complexes $D_{k\epsilon}\alpha = -\alpha_{e}$ imply certain constraints for the external 'currents' $\alpha_{e}$. Indeed, by the fact that
\[ \partial_{k-1} \circ \partial_{k} = 0 \]
it follows that
\[ D_{(k-1)e}\alpha_{e} = 0 \]
Furthermore, we can uniquely extend the open $k$-complex to an ordinary $k$-complex while keeping the same Betti numbers.
This means again that the external power can be rewritten as a product of 'external currents' and 'external voltages'.
Hamiltonian dynamics on a $k$-complex

There are a number of canonical ways to define 'physical' dynamics on $k$-complexes.

First option: define the Dirac structure

\[
\begin{align*}
  f_x &= -d_k f, \\ f_x &\in \Lambda^k, f \in \Lambda^{k-1} \\
  e &= \partial_k e_x, \\ e_x &\in \Lambda_k, e \in \Lambda_{k-1}
\end{align*}
\]
Associate to every $k$-cell an energy storage, leading to a total energy storage $H(x)$, where $x \in \Lambda^k$ denotes the vector of energy variables, with

$$
\dot{x} = -f_x, \quad e_x = \frac{\partial H}{\partial x}(x)
$$

Furthermore, associate to every $(k-1)$-cell the resistive relation

$$
f = -Re, \quad R = R^T \geq 0
$$

This yields the relaxation dynamics

$$
\dot{x} = -d_k e = d_k R f = -d_k R \partial_k \frac{\partial H}{\partial x}(x), \quad x \in \Lambda^k
$$

with the property that

$$
\frac{dH}{dt} = -\left(\partial_k \frac{\partial H}{\partial x}(x)\right)^T R \partial_k \frac{\partial H}{\partial x}(x) \leq 0
$$
For an open complex with external \((k-1)\)-cells and external 'currents' \(\bar{\Lambda} \subset \Lambda_{k-1}\) the definition is modified as follows.

Consider instead

\[
\begin{align*}
  f_x &= -d_k \begin{bmatrix} f \\ f_b \end{bmatrix}, & f_x \in \Lambda^k, & \begin{bmatrix} f \\ f_b \end{bmatrix} \in \Lambda^{k-1}, & f_b \in \bar{\Lambda}^{k-1} \\
  \begin{bmatrix} e \\ e_b \end{bmatrix} &= \partial_k e_x, & e_x \in \Lambda_k, & \begin{bmatrix} e \\ e_b \end{bmatrix} \in \Lambda_{k-1}, & f_b \in \bar{\Lambda}_{k-1}
\end{align*}
\]

(3)

with \(f_b, e_b\) corresponding to the external \((k-1)\)-cells, and \(f, e\) corresponding to the internal \((k-1)\)-cells.
Imposing the same storage relations $f_x = -\dot{x}, e_x = \frac{\partial H}{\partial x}(x)$ and resistive relations $f = -R e$ we arrive at

$$
\dot{x} = -d_k^r R \partial_k^r \frac{\partial H}{\partial x}(x) + d_k^b f_b \\
e_b = \partial_k^b \frac{\partial H}{\partial x}(x)
$$

where we have split $d_k$ as $d_k = \begin{bmatrix} d_k^r & d_k^b \end{bmatrix}$ and $\partial_k = \begin{bmatrix} \partial_k^r \\ \partial_k^b \end{bmatrix}$.

This defines a **port-Hamiltonian system** with inputs $f_b$ and outputs $e_b$. 
**Heat transfer on a 2-complex**

We will write the heat transfer as a conservation law in terms of the conservation of internal energy:

The internal energy $u$ of the 2-complex is a 2-cochain, $u \in \Lambda^2$ (with every component of $u$ denoting the energy of the corresponding 2-cell).

The thermodynamic properties are defined by Gibbs’ relation, and generated by the *entropy function* $s = s(u)$ as thermodynamic potential. Since we consider transformations with constant volume and without mass transfer, Gibbs’ relation reduces to the definition of the vector of intensive variables $e_u$, conjugated to the extensive variables $u$ by

$$e_u = \frac{\partial s}{\partial u}(u)$$

The components of the vector $e_u \in \Lambda_2$ are equal to the reciprocal of the temperature in each 2-cell.
The heat conduction is given by the *heat flux*

\[ f \in \Lambda^1 \]

describing the heat flux through every 1-cell (edge). This flux arises from thermal non-equilibrium, defined by the fact that the temperature is varying over the 2-cells.

Its conjugate vector of variables is the *thermodynamical driving force* vector

\[ e \in \Lambda_1 \]

given as the vector of the differences of the reciprocals of the temperatures of the 2-cells with common edges (1-cells)

\[ e = \partial_2 e_u \]
By Fourier’s law the heat flux due to thermal non-equilibrium is expressed as
\[ f = R(e_u) e, \]
with \( R(e_u) = R^T(e_u) \geq 0 \) depending on the heat conduction coefficients. (Note the sign-difference !). Finally
\[ \frac{du}{dt} = d_2 f \]

Hence the resulting system is a port-Hamiltonian system (of relaxation type) defined on the 2-complex, with vector of state variables \( x \) given by the internal energy vector \( u \), and Hamiltonian \( s(u) \).

By the different sign the entropy \( s(u) \) satisfies
\[ \frac{ds}{dt} = (\partial_2 \frac{\partial s}{\partial u}(u))^T R(e_u) \partial_2 \frac{\partial s}{\partial u}(u) \geq 0 \]
expressing the fact that the entropy $s(u)$ is monotonously increasing.

The exchange of heat through the boundary of the system can be incorporated by splitting the edges (1-cells) into internal edges with the resistive relation and external (boundary) edges. This would lead to

$$\frac{ds}{dt} = (\partial_2 \frac{\partial s}{\partial u}(u))^T R(e_u) \partial_2 \frac{\partial s}{\partial u}(u) + e_b f_b$$

with $f_b, e_b$ denoting the heat flux, respectively, thermodynamical driving force, through the boundary 1-cells.
Another type of Hamiltonian dynamics is obtained by considering the relations

\[
\begin{align*}
 f^1_x &= d_k e^x_2, \quad f^1_x \in \Lambda^k, \quad e^x_2 \in \Lambda^{k-1} \\
 f^2_x &= -\partial_k e^x_1, \quad e^x_1 \in \Lambda^k, \quad f^2_x \in \Lambda_{k-1}
\end{align*}
\]

defining again a Dirac structure, together with two energy-storage relations

\[
\begin{align*}
 \dot{x}^1 &= -f^1_x, \quad e^x_1 = \frac{\partial H^1}{\partial x^1}(x^1) \\
 \dot{x}^2 &= -f^2_x, \quad e^x_2 = \frac{\partial H^2}{\partial x^2}(x^2)
\end{align*}
\]

leading to the oscillatory dynamics

\[
\begin{align*}
 \dot{x}_1 &= -d_k \frac{\partial H}{\partial x^2}(x^1, x^2) \\
 \dot{x}_2 &= \partial_k \frac{\partial H}{\partial x^1}(x^1, x^2)
\end{align*}
\]

**Example 1:** \((k = 2)\) Discretized Maxwell's equations on a 2-dimensional domain.
**Example 2: Formation control**

Consider $n$ point masses

$$\dot{p}_i = u_i, \quad i = 1, \ldots, n$$

with $p_i$ denoting their momenta, and $v_i = \frac{p_i}{m_i}$ their velocities. Suppose $\bar{v}$ is a desired joint velocity vector, and moreover, we want their position vectors $q_i$ converge to a certain desired formation, e.g. (for $n = 3$)

$$|q_1 - q_2| = |q_2 - q_3| = |q_3 - q_1| = 1$$

This determines a graph structure with incidence matrix $D$.

---

\(^{a}\)M. Arcak, IEEE TAC, 52, August 2007
Apply feedback to obtain

\[ \dot{q}_i = \xi_i + \bar{v} \]
\[ \dot{\xi}_i = f_i \]

and use

\[ (\mathbf{D}^T \otimes \mathbf{I}_3)(\dot{q}) = (\mathbf{D}^T \otimes \mathbf{I}_3)(\xi) \]

as an input to a Hamiltonian integrator dynamics defined on the edges, leading to an input

\[ f := -(\mathbf{D} \otimes \mathbf{I}_3)(\psi) \]

for the dynamics at the nodes.

By designing carefully the Hamiltonian corresponding to the dynamics on the edges, and by adding damping, one obtains the desired formation control.
Model reduction of port-Hamiltonian systems

- Network modeling of complex lumped-parameter systems (circuits, multi-body systems) often leads to high-dimensional models.

- Structure-preserving spatial discretization of distributed-parameter port-Hamiltonian systems yields high-dimensional port-Hamiltonian models.

- Lumped-parameter modeling of systems like MEMS gives high-dimensional port-Hamiltonian systems.

- Controller systems may be in first instance distributed-parameter, and need to be discretized to low-order controllers.
In many cases we want the reduced-order system to be again port-Hamiltonian:

- Port-Hamiltonian model reduction preserves passivity.
- Port-Hamiltonian model reduction may (approximately) preserve other balance laws /conservation laws.
- Physical interpretation of reduced-order model.
- Reduced-order system can replace the high-order port-Hamiltonian system in a larger context.

Thus there is a need for structure-preserving model reduction of high-dimensional port-Hamiltonian systems.
**Controllability analysis**

Consider a linear port-Hamiltonian system, written as

\[
\dot{x} = FQx + Bu, \quad F := J - R, \quad J = -J^T, \quad R = R^T \geq 0 \\
y = B^T Qx, \quad Q = Q^T \geq 0
\]

Take linear coordinates \( x = (x_1, x_2) \) such that the upper part of

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
F_{11} & F_{12} \\
F_{21} & F_{22}
\end{bmatrix} \begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u
\]

\[
y = \begin{bmatrix}
B_1^T \\
B_2^T
\end{bmatrix} \begin{bmatrix}
Q_{11} & Q_{12} \\
Q_{21} & Q_{22}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

is the reachability subspace \( R \).
By invariance of $R$ this implies

$$F_{21}Q_{11} + F_{22}Q_{21} = 0$$

$$B_2 = 0$$

It follows that the dynamics restricted to $R$ is given as

$$\dot{x}_1 = (F_{11}Q_{11} + F_{12}Q_{21})x_1 + B_1u$$

$$y = B_1^TQ_{11}x_1$$

Now solve for $Q_{21}$ as $Q_{21} = -F_{22}^{-1}F_{21}Q_{11}$. This yields

$$\dot{x}_1 = (F_{11} - F_{12}F_{22}^{-1}F_{21})Q_{11}x_1 + B_1u$$

$$y = B_1^TQ_{11}x_1$$

which is again a port-Hamiltonian system.
Observability analysis

Suppose the system is not observable. Then there exist coordinates $x = (x_1, x_2)$ such that the lower part is the unobservability subspace $N$. By invariance of $N$ it follows that

$$F_{11}Q_{12} + F_{12}Q_{22} = 0$$
$$B_1^T Q_{12} + B_2^T Q_{22} = 0$$

Then the dynamics on the quotient space $\mathcal{X}/N$ is

$$\dot{x}_1 = (F_{11}Q_{11} + F_{12}Q_{21})x_1 + B_1 u$$
$$y = B_1^T Q_{11}x_1 + B_2^T Q_{21}x_1$$
It follow from that \( F_{12} = -F_{11}Q_{12}Q_{22}^{-1} \) and \( B_{2}^T = -B_{1}^TQ_{12}Q_{22}^{-1} \). Substitution yields

\[
\dot{x}_1 = F_{11}(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1 + B_1 u
\]
\[
y = B_1^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1
\]

which is again a port-Hamiltonian system with Hamiltonian

\[
\tilde{H} = \frac{1}{2}x_1^T(Q_{11} - Q_{12}Q_{22}^{-1}Q_{21})x_1.
\]

**Remark** Note that the Schur complement \((Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}) \geq 0\) if \(Q \geq 0\).

This suggests two canonical ways for structure-preserving model reduction.
Conclusions of Part I

- Port-Hamiltonian systems provide a unified framework for *modeling, analysis, and simulation* of complex lumped-parameter multi-physics systems.

- Inclusion of distributed-parameter components.

- Direct lumping of distributed-parameter systems to finite-dimensional port-Hamiltonian systems.

- Structure-preserving model reduction.
  - Extensions to thermodynamic systems and chemical reaction networks.
  - Further exploration of the network (graph) information.

See www.math.rug.nl/~arjan, for further info.
The port-Hamiltonian approach to physical system modeling and control

Part II: Control of Port-Hamiltonian systems

Contents

- Use of passivity for control
- Control by interconnection: set-point stabilization
- The dissipation obstacle
- A state feedback perspective; shaping the Hamiltonian
- New control paradigms
- Model reduction of port-Hamiltonian systems
Use of passivity for control and beyond

- The storage function can be used as Lyapunov function, implying some sort of stability for the uncontrolled system.

- The standard feedback interconnection of two passive systems is again passive, with storage function being the sum of the individual storage functions.

- Passive systems can be asymptotically stabilized by adding artificial damping. In fact,

\[ \frac{d}{dt} H \leq u^T y \]

together with the additional damping \( u = -y \) yields

\[ \frac{d}{dt} H \leq -\|y\|^2 \]

proving asymptotic stability provided an observability condition is met.
Example The Euler equations for the motion of a rigid body revolving about its center of gravity with one input are

\[ I_1 \dot{\omega}_1 = [I_2 - I_3] \omega_2 \omega_3 + g_1 u \]
\[ I_2 \dot{\omega}_2 = [I_3 - I_1] \omega_1 \omega_3 + g_2 u \]
\[ I_3 \dot{\omega}_3 = [I_1 - I_2] \omega_1 \omega_2 + g_3 u, \]

Here \( \omega := (\omega_1, \omega_2, \omega_3)^T \) are the angular velocities around the principal axes of the rigid body, and \( I_1, I_2, I_3 > 0 \) are the principal moments of inertia. The system for \( u = 0 \) has the origin as an equilibrium point. Linearization yields the linear system

\[
A = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \\
B = \begin{pmatrix}
I_1^{-1} g_1 \\
I_2^{-1} g_2 \\
I_3^{-1} g_3
\end{pmatrix}.
\]

Hence the linearization does not say anything about stabilizability.
Stability and asymptotic stabilization by damping injection

Rewrite the system in port-Hamiltonian form by defining the angular momenta

\[ p_1 = I_1 \dot{\omega}_1, \ p_2 = I_2 \dot{\omega}_2, \ p_3 = I_3 \dot{\omega}_3 \]

and defining the Hamiltonian \( H(p) \) as the total kinetic energy

\[ H(p) = \frac{1}{2} \left( \frac{p_1^2}{I_1} + \frac{p_2^2}{I_2} + \frac{p_3^2}{I_3} \right) \]

Then the system can be rewritten as

\[
\begin{bmatrix}
\dot{p}_1 \\
\dot{p}_2 \\
\dot{p}_3
\end{bmatrix} =
\begin{bmatrix}
0 & -p_3 & p_2 \\
p_3 & 0 & -p_1 \\
-p_2 & p_1 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial p_1} \\
\frac{\partial H}{\partial p_2} \\
\frac{\partial H}{\partial p_3}
\end{bmatrix}
+ \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} u,
\]

\[ y = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \]
Since $\dot{H} = 0$ and $H$ has a minimum at $p = 0$ the origin is **stable**. Damping injection amounts to the negative output feedback

$$u = -y = -g_1 \frac{p_1}{I_1} - g_2 \frac{p_2}{I_2} - g_3 \frac{p_3}{I_3} = -g_1 \omega_1 - g_2 \omega_2 - g_3 \omega_3,$$

yielding convergence to the largest invariant set contained in

$$S := \{ p \in \mathbb{R}^3 \mid \dot{H}(p) = 0 \} = \{ p \in \mathbb{R}^3 \mid g_1 \frac{p_1}{I_1} + g_2 \frac{p_2}{I_2} + g_3 \frac{p_3}{I_3} = 0 \}$$

It can be shown that the largest invariant set contained in $S$ is the origin $p = 0$ if and only if

$$g_1 \neq 0, g_2 \neq 0, g_3 \neq 0,$$

in which case the origin is rendered **asymptotically stable** (even, globally).
Beyond control via passivity: What can we do if the desired set-point is not a minimum of the storage function??

Recall the proof of stability of an equilibrium \((\omega_1^*, 0, 0) \neq (0, 0, 0)\) of the Euler equations.

The total energy \(H = \frac{2I_1}{p_1^2} + \frac{2I_2}{p_2^2} + \frac{2I_3}{p_3^2} = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2\) has a minimum at \((0, 0, 0)\). Stability of \((\omega_1^*, 0, 0)\) is shown by taking as Lyapunov function a combination of the total energy \(K\) and another conserved quantity, namely the total angular momentum 

\[ C = p_1^2 + p_2^2 + p_3^2 = I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2 \]

This follows from

\[
\begin{bmatrix}
  p_1 & p_2 & p_3 \\
  p_2 & p_1 & 0 \\
  p_3 & 0 & -p_1 \\
\end{bmatrix}
\begin{bmatrix}
  0 & -p_3 & p_2 \\
  p_3 & 0 & -p_1 \\
  -p_2 & p_1 & 0 \\
\end{bmatrix} = 0
\]
In general, for any Hamiltonian dynamics

\[ \dot{x} = J(x) \frac{\partial H}{\partial x}(x) \]

one may search for conserved quantities \( C \), called \textbf{Casimirs}, as being solutions of

\[ \frac{\partial^T C}{\partial x}(x)J(x) = 0 \]

Then \( \frac{d}{dt} C = 0 \) for every \( H \), and thus also \( H + C \) is a \textbf{candidate Lyapunov function}.

Note that the minimum of \( H + C \) may now be \textbf{different} from the minimum of \( H \).
Control by interconnection: set-point stabilization:

Consider first a lossless Hamiltonian plant system $P$

$$\dot{x} = J(x) \frac{\partial H}{\partial x}(x) + g(x)u$$

$$y = g^T(x) \frac{\partial H}{\partial x}(x)$$

where the desired set-point $x^*$ is not a minimum of the Hamiltonian $H$, while the Hamiltonian dynamics $\dot{x} = J(x) \frac{\partial H}{\partial x}(x)$ does not possess useful Casimirs.

How to (asymptotically) stabilize $x^*$?
**Control by interconnection:**

Consider a *controller* port-Hamiltonian system

\[
\frac{d\xi}{dt} = J_c(\xi) \frac{\partial H_c}{\partial \xi}(\xi) + g_c(\xi)u_c, \quad \xi \in \mathcal{X}_c
\]

\[
C : \quad y_c = g^T(\xi) \frac{\partial H_c}{\partial \xi}(\xi)
\]

via the standard feedback interconnection

\[
u = -y_c, \quad u_c = y
\]
Then the closed-loop system is the port-Hamiltonian system

\[
\begin{bmatrix}
\dot{x} \\
\dot{\xi}
\end{bmatrix} =
\begin{bmatrix}
J(x) & -g(x)g_c^T(\xi) \\
g_c(\xi)g^T(x) & J_c(\xi)
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H(x)}{\partial x} \\
\frac{\partial H_c(\xi)}{\partial \xi}
\end{bmatrix}
\]

with state space $\mathcal{X} \times \mathcal{X}_c$, and total Hamiltonian $H(x) + H_c(\xi)$.

**Main idea:** design the controller system in such a manner that the closed-loop system has useful Casimirs $C(x, \xi)$!

This may lead to a suitable candidate Lyapunov function

\[
V(x, \xi) := H(x) + H_c(\xi) + C(x, \xi)
\]

with $H_c$ to-be-determined.
Thus we look for functions $C(x, \xi)$ satisfying

$$\begin{bmatrix}
\frac{\partial^T C}{\partial x}(x, \xi) & \frac{\partial^T C}{\partial \xi}(x, \xi)
\end{bmatrix}
\begin{bmatrix}
J(x) & -g(x)g_c^T(\xi) \\
g_c(\xi)g^T(x) & J_c(\xi)
\end{bmatrix} = 0$$

such that the candidate Lyapunov function

$$V(x, \xi) := H(x) + H_c(\xi) + C(x, \xi)$$

has a minimum at $(x^*, \xi^*)$ for some (or a set of) $\xi^* \Rightarrow$ stability.

**Remark:** The set of such achievable closed-loop Casimirs $C(x, \xi)$ can be fully characterized.

Subsequently, one may add extra damping (directly or in the dynamics of the controller) to achieve asymptotic stability.
Example: the ubiquitous pendulum

Consider the mathematical pendulum with Hamiltonian

\[ H(q, p) = \frac{1}{2}p^2 + (1 - \cos q) \]

actuated by a torque \( u \), with output \( y = p \) (angular velocity). Suppose we wish to stabilize the pendulum at a non-zero angle \( q^* \) and \( p^* = 0 \).

Apply the nonlinear integral control

\[ \dot{\xi} = u_c = y \]
\[ -u = y_c = \frac{\partial H_c}{\partial \xi}(\xi) \]

which is a port-Hamiltonian controller system with \( J_c = 0 \).
Casimirs $C(q, p, \xi)$ are found by solving

\[
\begin{bmatrix}
\frac{\partial C}{\partial q} & \frac{\partial C}{\partial p} & \frac{\partial C}{\partial \xi}
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & -1 \\
0 & 1 & 0
\end{bmatrix}
= 0
\]

leading to Casimirs $C(q, p, \xi) = K(q - \xi)$, and candidate Lyapunov functions

\[
V(q, p, \xi) = \frac{1}{2} p^2 + (1 - \cos q) + H_c(\xi) + K(q - \xi)
\]

with the functions $H_c$ and $K$ to be determined.
For a local minimum, determine $K$ and $H_c$ such that

**Equilibrium assignment**

\[
\sin q^* + \frac{\partial K}{\partial z} (q^* - \xi^*) = 0
\]
\[
-\frac{\partial K}{\partial z} (q^* - \xi^*) + \frac{\partial H_c}{\partial \xi} (\xi^*) = 0
\]

**Minimum condition**

\[
\begin{bmatrix}
\cos q^* + \frac{\partial^2 K}{\partial z^2} (q^* - \xi^*) & 0 & -\frac{\partial^2 K}{\partial z^2} (q^* - \xi^*) \\
0 & 1 & 0 \\
-\frac{\partial^2 K}{\partial z^2} (q^* - \xi^*) & 0 & \frac{\partial^2 K}{\partial z^2} (q^* - \xi^*) + \frac{\partial^2 H_c}{\partial \xi^2} (\xi^*)
\end{bmatrix} > 0
\]

Many possible solutions.
Example: stabilization of the shallow water equations

The dynamics of the water in a canal can be described by

\[
\begin{bmatrix} \partial_t h \\ \partial_z h \end{bmatrix} + \begin{bmatrix} v & h \\ g & v \end{bmatrix} \begin{bmatrix} v \\ v \end{bmatrix} = 0
\]

with \( h(z,t) \) the height of the water at position \( z \), and \( v(z,t) \) its velocity (and \( g \) the gravitational constant).

Recall that by recognizing the total energy

\[
H(h,v) = \int_a^b H dz = \int_a^b \frac{1}{2} [hv^2 + gh^2] dz
\]

this can be written (similarly to the telegrapher’s equations) as the port-Hamiltonian system
\[
\frac{\partial h}{\partial t}(z,t) = -\frac{\partial}{\partial z} \frac{\partial H}{\partial v}(h,v)
\]

\[
\frac{\partial v}{\partial t}(z,t) = -\frac{\partial}{\partial z} \frac{\partial H}{\partial h}(h,v)
\]

with the 4 boundary variables

\[
hv|_{a,b} - \left( \frac{1}{2} v^2 + gh \right)|_{a,b}
\]

denoting respectively the **mass flow** and the **Bernoulli function** at the boundary points \(a, b\).

(Note that the product \(hv \cdot (\frac{1}{2} v^2 + gh)\) equals **power**.)
Suppose we want to control the water level $h$ to a desired height $h^*$. 

An obvious 'physical' controller is to add to one side of the canal, say the right-end $b$, an infinite water reservoir of height $h^*$, corresponding to the port-Hamiltonian 'source' system

$$
\dot{\xi} = u_c \\
y_c = \frac{\partial H_c}{\partial \xi} (= gh^*)
$$

with Hamiltonian $H_c(\xi) = gh^*\xi$, by the feedback interconnection

$$
u_c = y = h(b)v(b), \quad y_c = -u = \frac{1}{2}v^2(b) + gh(b)
$$

This yields a closed-loop port-Hamiltonian system with total Hamiltonian

$$\int_0^l \frac{1}{2}[hv^2 + gh^2]dz + gh^*\xi$$
By mass balance, 
\[ \int_{a}^{b} h(z, t) \, dz + \xi + c \]
is a Casimir for the closed-loop system. Thus we may take as Lyapunov function 
\[ V(h, v, \xi) := \frac{1}{2} \int_{a}^{b} [hv^2 + gh^2] \, dz + gh^* \xi - gh^* [\int_{a}^{b} h(z, t) \, dz + \xi] + \frac{1}{2} g(b - a)h^{*2} \]
\[ = \frac{1}{2} \int_{a}^{b} [hv^2 + g(h - h^*)^2] \, dz \]
which has a minimum at the desired set-point \((h^*, v^* = 0, \xi^*)\) (with \(\xi^*\) arbitrary).

**Remark** Note that the source port-Hamiltonian system is **not** passive, since the Hamiltonian \(H_c(\xi) = gh^*\xi\) is not bounded from below.
An alternative, passive, choice of the Hamiltonian controller system is to take e.g.

\[ H_c(\xi) = \frac{1}{2} gh^* \xi^2 \]

leading to the Lyapunov function

\[ V(h, v, \xi) = \frac{1}{2} \int_a^b \left[ hv^2 + g(h - h^*)^2 \right] dz + \frac{1}{2} gh^*(\xi - 1)^2 \]

**Asymptotic stability** of the equilibrium \((h^*, v^* = 0, \xi^* = 1)\) can be obtained by adding 'damping', that is, replacing \(u_c = y = h(b)v(b)\) by

\[ u_c := y - \frac{\partial V}{\partial \xi}(\xi) = h(b)v(b) - gh^*(\xi - 1) \]

leading to (if there is no power flow through the left-end \(a\))

\[ \frac{d}{dt} V = -gh^*(\xi - 1)^2 \]

(See also the work of Bastin & co-workers for related and more refined results.)
The dissipation obstacle

Surprisingly, the presence of dissipation $R \neq 0$ may pose a problem! $C(x)$ is a Casimir for the Hamiltonian dynamics with dissipation

$$
\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x), \quad J = J^T, R = R^T \geq 0
$$

iff

$$
\frac{\partial^T C}{\partial x}[J - R] = 0 \Rightarrow \frac{\partial^T C}{\partial x}[J - R] \frac{\partial C}{\partial x} = 0 \Rightarrow \frac{\partial^T C}{\partial x} R \frac{\partial C}{\partial x} = 0 \Rightarrow \frac{\partial^T C}{\partial x} R = 0
$$

and thus $C$ is a Casimir iff

$$
\frac{\partial^T C}{\partial x}(x)J(x) = 0, \quad \frac{\partial^T C}{\partial x}(x)R(x) = 0
$$

The physical reason for the dissipation obstacle is that by using a passive controller only equilibria where no energy-dissipation takes place may be stabilized.
Similarly, if $C(x, \xi)$ is a Casimir for the closed-loop port-Hamiltonian system then it must satisfy

$$\begin{bmatrix}
\frac{\partial^T C}{\partial x}(x, \xi) & \frac{\partial^T C}{\partial \xi}(x, \xi)
\end{bmatrix}
\begin{bmatrix}
R(x) & 0 \\
0 & R_c(\xi)
\end{bmatrix} = 0$$

implying by semi-positivity of $R(x)$ and $R_c(x)$

$$\frac{\partial^T C}{\partial x}(x, \xi) R(x) = 0$$

$$\frac{\partial^T C}{\partial \xi}(x, \xi) R_c(\xi) = 0$$

This is the **dissipation obstacle**, which implies that one cannot shape the Lyapunov function in the coordinates that are directly affected by energy dissipation.

**Remark**: For shaping the potential energy in mechanical systems this is **not** a problem since dissipation enters in the differential equations for the momenta.
To overcome the dissipation obstacle

Suppose one can find a mapping $C : \mathcal{X} \to \mathbb{R}^m$, with its (transposed) Jacobian matrix $K^T(x) := \frac{\partial C}{\partial x}(x)$ satisfying

$$[J(x) - R(x)]K(x) + g(x) = 0$$

**Construct** now the interconnection and dissipation matrix of an augmented system as

$$J_{aug} := \begin{bmatrix} J & JK \\ KTJ & KTJK \end{bmatrix}, \quad R_{aug} := \begin{bmatrix} R & RK \\ KT R & KT RK \end{bmatrix}$$

By construction

$$[K^T(x) \mid -I]J_{aug} = [K^T(x) \mid -I]R_{aug} = 0$$

implying that the components of $C$ are Casimirs for the Hamiltonian dynamics
\[
\begin{bmatrix}
\dot{x} \\
\dot{\xi}
\end{bmatrix} = [J_{aug} - R_{aug}] \begin{bmatrix}
\frac{\partial H}{\partial x}(x) \\
\frac{\partial H_c}{\partial \xi}(\xi)
\end{bmatrix}
\]

Furthermore, since 
\[
[J(x) - R(x)]K(x) + g(x) = 0
\]

Moreover, since
\[
J_{aug} - R_{aug} =
\begin{bmatrix}
J - R & [J - R]K \\
K^T[J - R] & K^TJK - K^TRK
\end{bmatrix}
\]

\[
= \begin{bmatrix}
J - R & -g \\
[g - 2RK]^T & K^TJK - K^TRK
\end{bmatrix}
\]

Thus the augmented system is a closed-loop system for a different output!
Port-Hamiltonian systems with **feedthrough term** take the form

\[
\dot{x} = [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u \\
y = (g(x) + 2P(x))^T \frac{\partial H}{\partial x}(x) + [M(x) + S(x)]u,
\]

with \(M\) skew-symmetric and \(S\) symmetric, while

\[
\begin{bmatrix}
R(x) & P(x) \\
P^T(x) & S(x)
\end{bmatrix} \geq 0
\]
The augmented system is thus the feedback interconnection of the nonlinear integral controller

\[
\dot{\xi} = u_c \\
y_c = \frac{\partial H_c}{\partial \xi}(\xi)
\]

with the plant port-Hamiltonian system with modified output with feedthrough term

\[
\dot{x} = [J(x) - R(x)]\frac{\partial H}{\partial x}(x) + g(x)u
\]

\[
y_{mod} = [g(x) - 2R(x)K(x)]^T\frac{\partial H}{\partial x}(x) + [-K^T(x)J(x)K(x) + K^T(x)R(x)K(x)]u
\]

Remark: See Jeltsema, Ortega and Scherpen for further explorations.
Generalization to feedback interconnection with state-modulation.

Recall that $K^T(x) := \frac{\partial C}{\partial x}(x)$ is a solution to
$[J(x) - R(x)]K(x) + g(x) = 0$. This can be generalized to

$$[J(x) - R(x)]K(x) + g(x)\beta(x) = 0$$

with $\beta(x)$ an $m \times m$ design matrix.

The same scheme as above works if we extend the standard feedback interconnection $u = -y_c, u_c = y$ to the state-modulated feedback

$$u = -\beta(x)y_c, \quad u_c = \beta^T(x)y$$

Note that $K(x)$ is a solution for some $\beta(x)$ iff

$$g^\perp(x)[J(x) - R(x)]K(x) = 0$$

(In fact, $\beta(x) := -(g^T(x)g(x))^{-1}g^T(x)[J(x) - R(x)]K(x)$ does the job.)
A state feedback perspective: shaping the Hamiltonian

Restrict (without much loss of generality) to Casimirs of the form

\[ C(x, \xi) = \xi_j - G_j(x) \]

It follows that for all time instants

\[ \xi_j = G_j(x) + c_j, \quad c_j \in \mathbb{R} \]

Suppose that in this way all control state components \( \xi_i \) can be expressed as function

\[ \xi = G(x) \]

of the plant state \( x \). Then the dynamic feedback reduces to a state feedback, and the Lyapunov function \( H(x) + H_c(\xi) + C(x, \xi) \) reduces to the shaped Hamiltonian

\[ H(x) + H_c(G(x)) \]
A direct state feedback perspective: 
Interconnection-Damping Assignment (IDA)-PBC control

A direct way to generate candidate Lyapunov functions $H_d$ is to look for state feedbacks $u = \hat{u}_{IDA}(x)$ such that

$$
\left[ J(x) - R(x) \right] \frac{\partial H}{\partial x}(x) + g(x)\hat{u}_{IDA}(x) = \left[ J_d(x) - R_d(x) \right] \frac{\partial H_d}{\partial x}(x)
$$

where $J_d$ and $R_d$ are newly assigned interconnection and damping structures.

Remark: For mechanical systems IDA-PBC control is equivalent to the theory of Controlled Lagrangians (Bloch, Leonard, Marsden, .).
For $J_d = J$ and $R_d = R$ (Basic IDA-PBC) this reduces to

$$[J(x) - R(x)] \frac{\partial(H_d - H)}{\partial x}(x) = g(x)\hat{u}_{BIDA}(x)$$

and thus in this case, there exists an $\hat{u}_{BIDA}(x)$ if and only if

$$g^\perp(x)[J(x) - R(x)] \frac{\partial(H_d - H)}{\partial x}(x) = 0$$

which is the same equation as obtained for stabilization by Casimir generation with a state-modulated nonlinear integral controller!

**Conclusion**: Basic IDA-PBC $\Leftrightarrow$ State-modulated Control by Interconnection.
**Shifted passivity w.r.t. a controlled equilibrium**

(see Jayawardhana, Ortega). Consider a port-Hamiltonian system

\[
\begin{align*}
\dot{x} &= Fz + gu, \quad z = \frac{\partial H}{\partial x}(x) \\
y &= g^Tz
\end{align*}
\]

where \( F = J - R, g \) are constant, and a controlled equilibrium \( x_0 \):

\[
Fz_0 + gu_0 = 0, \quad z_0 = \frac{\partial H}{\partial x}(x_0)
\]

Define the shifted storage function

\[
V(x) := H_p(x) - (x - x_0)^T \frac{\partial H_p}{\partial x}(x_0) - H_p(x_0)
\]

Note that \( \frac{\partial V}{\partial x} = z - z_0 \). It follows that

\[
\begin{align*}
\frac{d}{dt}V &= (z - z_0)^T \dot{x} = (z - z_0)^T(Fz + gu) = \\
&= (z - z_0)^TF(z - z_0) + (z - z_0)^Tg(u - u_0) + (z - z_0)^T(Fz_0 + gu_0) \leq (y - y_0)^T(u - u_0)
\end{align*}
\]

implying passivity w.r.t. the shifted inputs \( u - u_0 \) and outputs \( y - y_0 \).
Application to switching control

Consider the port-Hamiltonian model of a power-converter

\[ \dot{x} = F(\rho)z + g(\rho)E + g_l u, \quad z = \frac{\partial H_p}{\partial x}(x), \ F(\rho) := J(\rho) - R(\rho) \]

with vector of Boolean variables \( \rho \in \{0, 1\}^k \), \( H_p(x) \) the total stored electromagnetic energy, and output vector \( y = g_l^T z \).

Let \( x_0 \) be an equilibrium of the averaged model, that is

\[ F(\rho_0)z_0 + g(\rho_0)E + g_l u_0 = 0, \quad z_0 = \frac{\partial H}{\partial x}(x_0) \]

for some \( \rho_0 \in [0, 1]^k \) and \( u_0 \). Then

\[
\begin{align*}
\dot{x} & = F(\rho)(z - z_0) + F(\rho)z_0 + g(\rho)E + g_l u \\
& = F(\rho)(z - z_0) + [F(\rho) - F(\rho_0)]z_0 + [g(\rho) - g(\rho_0)]E + g_l(u - u_0) \\
& \quad + F(\rho_0)z_0 + g(\rho_0)E + g_l u_0 \\
& = F(\rho)(z - z_0) + [F(\rho) - F(\rho_0)]z_0 + [g(\rho) - g(\rho_0)]E + g_l(u - u_0)
\end{align*}
\]
For many power converters we know that

\[ F(\rho) - F(\rho_0) = \sum_{i=1}^{p} F_i(\rho_i - \rho_{0i}) \]

\[ g(\rho) - g(\rho_0) = \sum_{i=1}^{p} g_i(\rho_i - \rho_{0i}) \]

and thus

\[ \dot{x} = F(\rho)(z - z_0) + \sum_{i=1}^{p} [F_i z_0 + g_i E](\rho_i - \rho_{0i}) + g_l(u - u_0) \]

Take as Lyapunov/storage function

\[ V(x) := H_p(x) - (x - x_0)^T \frac{\partial H_p}{\partial x}(x_0) - H_p(x_0) \]

Then

\[ \frac{d}{dt} V(x) = [\frac{\partial H_p}{\partial x}(x) - \frac{\partial H_p}{\partial x}(x_0)]^T \dot{x} = (z - z_0)^T \dot{x} = \]

\[ (z - z_0)^T F(\rho)(z - z_0) + \sum_{i=1}^{p} (z - z_0)^T [F_i z_0 + g_i E](\rho_i - \rho_{0i}) + (z - z_0)^T g_l(u - u_0) \]

with \((z - z_0)^T F(\rho)(z - z_0) \leq 0\).
Thus at any time we can choose \( \rho_i \in \{0, 1\} \) such that

\[
\frac{d}{dt} V(x) \leq (z - z_0)^T g_l (u - u_0)
\]

implying passivity of the switched system with respect to the input vector \( u - u_0 \) and output vector \( y - y_0 = g_l^T (z - z_0) \). As a consequence, if the converter is terminated on a static resistive load then the switched converter is (asymptotically) stable around \( x_0 \). Thus the voltage over the resistive load can be stabilized around any set-point.

This can be immediately generalized to converters **connected to a load via a transmission line** (see Part I).

Note that for linear capacitors and inductors we have

\[
H_p(x) = \frac{1}{2} x^T Q x, \quad V(x) = \frac{1}{2} (x - x_0)^T Q (x - x_0)
\]

(cf. Buisson & co-workers)
New control paradigms

Example: Energy transfer control

Consider two port-Hamiltonian systems $\Sigma_i$

$$\dot{x}_i = J_i(x_i) \frac{\partial H_i}{\partial x_i}(x_i) + g_i(x_i) u_i$$

$$y_i = g^T_i(x_i) \frac{\partial H_i}{\partial x_i}(x_i), \quad i = 1, 2$$

Suppose we want to transfer the energy from the port-Hamiltonian system $\Sigma_1$ to the port-Hamiltonian system $\Sigma_2$, while keeping the total energy $H_1 + H_2$ constant.
This can be done by using the output feedback

\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} =
\begin{bmatrix}
  0 & -y_1 y_2^T \\
  y_2 y_1^T & 0
\end{bmatrix}
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix}
\]

It follows that the closed-loop system is energy-preserving. However, for the individual energies

\[
\frac{d}{dt} H_1 = -y_1^T y_1 y_2^T y_2 = -||y_1||^2 ||y_2||^2 \leq 0
\]

implying that \( H_1 \) is decreasing as long as \( ||y_1|| \) and \( ||y_2|| \) are different from 0. On the other hand,

\[
\frac{d}{dt} H_2 = y_2^T y_2 y_1^T y_1 = ||y_2||^2 ||y_1||^2 \geq 0
\]

implying that \( H_2 \) is increasing at the same rate. Has been successfully applied to energy-efficient path-following control of mechanical systems (cf. Duindam & Stramigioli).
Impedance control

Consider a system with two (not necessarily distinct) ports

\[
\begin{align*}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u + k(x)f, \quad x \in \mathcal{X}, u \in \mathbb{R}^m \\
y &= g^T(x) \frac{\partial H}{\partial x}(x) \quad u, y \in \mathbb{R}^m \\
e &= k^T(x) \frac{\partial H}{\partial x}(x) \quad f, e \in \mathbb{R}^m 
\end{align*}
\] (4)

The relation between the \(f\) and \(e\) variables is called the 'impedance' of the \((f,e)\)-port. In **Impedance Control** (Hogan) one tries to *shape* this impedance by using the control port corresponding to \(u, y\).

**Typical application:** the \((f, e)\)-port corresponds to the end-point of a robotic manipulator, while the \((u, y)\)-port corresponds to actuation.

**Basic question:** what are achievable impedances of the \((f, e)\)-port?
Conclusions of Part II

- Beyond passivity by port-Hamiltonian systems theory.
- Control by interconnection and Casimir generation, IDA-PBC control.
- Allows for 'physical' interpretation of control strategies. Suggests new control paradigms for nonlinear systems.
  - Use of passivity generally yields good robustness, but performance theory is yet lacking.
THANK YOU!